Classical tests of GTR

Mercury perihelion advance.

In the 19th century it was discovered that interplantary perturbations cannot account fully for the turning rate of the Mercury's orbit. About 43 arcseconds per century remained unexplained. The general theory of relativity exactly accounts for this descrepancy.

The newtonian equation for the trajectory of a planet, $u'' + u = \frac{M}{J^2}$, (where $u = 1/r$ and $u' = \frac{du}{d\phi}$) has a periodic elliptic solution $u = A\sin(\phi - \phi_0) + \frac{M}{J^2}$ with an angular period of 2π. The corresponding relativistic equation,

$$u'' + u = \frac{M}{J^2} + 3Mu^2 \ , \tag{1}$$

has an additional relativistic term $3Mu^2$ which causes the perihelion to shift.

Let us try to find a correction ϵ to the angular frequency by searching for a solution in the form $u = A\sin(1+\epsilon)\phi + B$. Setting this into the equation and collecting terms with $\sin(1+\epsilon)\phi$ gives

$$-A2\epsilon\sin(1+\epsilon)\phi = 3MA2B\sin(1+\epsilon)\phi \ , \tag{2}$$

which gives $\epsilon = -\frac{3M^2}{J^2}$ and correspondingly the shift of the orbit, $\Delta\phi = 2\pi\frac{3M^2}{J^2}$. This accounts precisely for the unexplained advance of the orbit.

Bending of light.

General relativity predicts apparent bending of light rays passing through gravitational fields. The bending was first observed in 1919 by A.S. Eddington during a total eclipse when stellar images near the occulted disk of the Sun appeared displaced by some arcseconds from their usual locations in the sky.

In the newtonian theory the light rays travel along straight lines described by the equation $u'' + u = 0$ with the straight-line solution $u = A\sin(\phi - \phi_0)$. The corresponding relativistic equation

$$u'' + u = 3Mu^2 \tag{3}$$

has an additional term, which causes the light trajectory to deflect from the straight line. Searching for the solution in the form $u = A\cos\phi + \epsilon(\phi)$, where $\epsilon(\phi)$ is a small correction, gives $\epsilon'' + \epsilon = 3MA^2\cos^2\phi$. Assuming $\epsilon(\phi) = C\cos^2\phi + D$ gives $\epsilon = MA^2(2 - \cos^2\phi)$. The incoming and outgoing rays ($r = \infty$) correspond to the angles ϕ_0 which are the solutions to the equations $u(\phi_0) = 0$. Searching for the solution perturbatively in the form $\phi_0 = \pi/2 + \delta\phi$ gives $\delta\phi = 2MA$.

Thus the angle of deflection between the in-going and out-going rays is $\Delta\phi = 2\delta\phi = 4MA = \frac{4M}{r_0}$ where r_0 is the closest distance between the ray and the central body.

Gravitational redshift.

Gravitational red shift is a change of the frequency of the electro-magnetic radiation as it passes through a gravitational field. It is a direct consequence of the equivalence principle.

The connection between the proper time interval $\Delta\tau$ and the world time interval Δt (here we only consider stationary gravitational fields where such world time can be introduced) is $\Delta\tau = \sqrt{g_{00}}\Delta t$.

Since frequencies are inversely proportional to the time intervals the corresponding connection between world frequency ω_0 and the locally measured frequency ω is $\omega = \frac{\omega_0}{\sqrt{g_{00}}}$. In a weak gravitational field $g_{00} = 1 + 2\phi$ and therefore $\omega = \omega_0(1 - \phi)$. A photon emitted from a point with ϕ_1 and received at a point with ϕ_2 will be shifted by $\Delta\omega = (\phi_1 - \phi_2)\omega$.

The famous experiment which verified the gravitational redshift is generally called the Pound-Rebka-Snider experiment where Mossbauer effect was used to accurately meauser the change of frequency of a photon travelling upwards 22 m in the Earth's field.

Exercises

1. Derive the Kepler's law (the relation between the orbit's period and the radius) for a circular orbit in Schwarzschild metric[1]. Hint: period$=2\pi/\omega$, where $\omega = d\phi/dt$ is the angular frequency which can be found from the geodesics $Du^r = 0$.

2. Show that in a synchronous reference frame ($ds^2 = d\tau^2 + g_{\alpha\beta}dx^\alpha dx^\beta$, where $\alpha, \beta = 1, 2, 3$) the time lines are geodesics.

3. Gravitational waves. In a weak gravitational field the metric tensor g_{ab} is equal to the flat metric η_{ab} plus a small term $h_{ab} : g_{ab} = \eta_{ab} + h_{ab}$. Show the the Riemann tensor to the lowest order in h_{ab} is

$$R_{abcd} = \frac{1}{2}(h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac}).$$

Show, that if coordinates satisfy the condition[2] $(h^a_b - \frac{1}{2}h\delta^a_b)^{;b} = 0$, the Ricci tensor is simplifies to $R_{ab} = -\frac{1}{2}h_{ab,c}^{;c}$. Show that the vacuum Einstein equation now turns into the ordinary wave equation $(\frac{\partial^2}{\partial t^2} - \Delta)h_{ab} = 0$.

[1] Answer: like in Newtonian theory, $\omega^2 = M/r^3$.
[2] $h \equiv h^a_a$