GTR FAQ

Covariant differentiation

Is the differential of a covariant scalar, $d\varphi$, a covariant scalar?
Yes, it is.

Is the differential of a covariant vector, dA^a, a covariant vector?
Only in Minkowski space, where $dg_{ab} = 0$. Generally dA_a is not a covariant vector, since in curvilinear coordinates $dA_a = d(g_{ab}A^b) \neq g_{ab}dA^b$.

Is the covariant differential of a covariant vector, DA^a, a covariant vector?
Yes, it is, since $DA_a = g_{ab}DA^b = D(g_{ab}A^b)$.

What are the definitions of ∂_a, D_a, $_{,a}$, and $_{;a}$?
$\partial_a f \equiv f_{,a} \equiv \frac{df}{dx^a}$, $D_a f \equiv f_{;a} \equiv \frac{Df}{dx^a}$

Is D_a a covariant thing?
Yes. In particular $D_a\varphi$ is a covariant vector, D_aA_b is covariant tensor and so forth.

Is ∂_a a covariant thing?
Only in Minkowski space. Generally, although $\partial_a\varphi$ is indeed a covariant vector, ∂_aA_b is NOT a covariant tensor. However, the antisymmetric combination $\partial_aA_b - \partial_bA_a = D_aA_b - D_bA_a$ is a covariant tensor.

What is the definition of D^a?
Since D_a is a covariant operator, the index in raised in the usual way, $D^a = g^{ab}D_b = D_bg^{ba}$.

What is the definition of ∂^a?
Generally, since ∂_a is not a covariant operator, one cannot raise the index and therefore ∂^a is not defined. However, in Minkowski space, where the metric tensor is constant, $g_{ab} = \eta_{ab}$, this thing is defined as $\partial^a = \eta^{ab}\partial_b = \partial_b\eta^{ba}$.

What is the definition of the electromagentic tensor F_{ab} in curvilinear coordinates?
The electromagentic tensor F_{ab} is defined in a covariant way as $F_{ab} = D_aA_b - D_bA_a \equiv A_{b;a} - A_{a;b}$. However this particular combination can also be written with ordinary derivatives, $D_aA_b - D_bA_a = \partial_aA_b - \partial_bA_a \equiv A_{b,a} - A_{a,b}$.

And then what is $F^a_{\ b}$ and F^{ab}?
Since F_{ab} is a covariant tensor, the indexes are raised in the usual way, $F^a_{\ b} = g^{ac}F_{cb}$, $F^{ab} = g^{ac}F^a_{\ c}$.

But isn't $F^{ab} = \partial^aA^b - \partial^bA^a$?
Only in Minkowski space. Generally, since ∂^a is not defined, the tensor F^{ab} cannot be written this way in curvilinear coordinates.