General relativity: note9

Schwarzschild metric is a static spherically symmetric solution of the vacuum Einstein's equations, $R_{ab} = 0$:

$$ds^2 = (1 - \frac{r_g}{r})dt^2 - (1 - \frac{r_g}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where $r_g = 2Gmc^{-2}$ is the gravitational (Schwarzschild) radius of the central body with mass m.

Radiall fall in the Schwarzshild field. Lemaitre coordinates. Event horizons. Black holes.

In the Schwarzschild metric the gravitational radius $r_g = 2m$ is a singular point. Inside the Schwarzschild radius time and radial coordinates interchange. A transformation to the new coordinates τ, ρ

$$d\tau = dt + \sqrt{\frac{r_g}{r}}\frac{1}{(1-\frac{r_g}{r})}dr \ , \quad d\rho = dt + \sqrt{\frac{r}{r_g}}\frac{1}{(1-\frac{r_g}{r})}dr \tag{2}$$

leads to the Lemaitre metric

$$ds^2 = d\tau^2 - \frac{r_g}{r}d\rho^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \quad \text{where } \mathrm{r} = [\frac{3}{2}(\rho - \tau)]^{2/3}\mathrm{r_g^{1/3}} \tag{3}$$

which is related to free particles radially falling towards the center.

For a free falling body, $d\rho = 0$, in the region $r \approx r_g$

$$dt \approx -\frac{r_g}{r - r_g}dr \ , \quad \Rightarrow \ r - r_g = (r_0 - r_g)\exp\left(-\frac{t - t_0}{r_g}\right), \tag{4}$$

it takes a free falling body infinitely long time to reach the Schwarzschild radius (for the outer observer).

In τ, ρ coordinates the free falling particle reaches the Schwarzschild radius and the origin within finite time

$$\int_{r_0}^{r_g} d\tau = \frac{2}{3}\left(\frac{r_0^{3/2} - r_g^{3/2}}{r_g^{1/2}}\right) \tag{5}$$

Along the tragectory of a (radial) light ray

$$dr = \sqrt{\frac{r}{r_g}}\left(\pm\sqrt{\frac{r}{r_g}} - 1\right)d\tau, \tag{6}$$

therefore no signal can escape from inside the Schwarzschild radius where always $dr < 0$ and the light rays emitted radially inwards and outwards both end up at the origin.

Excercises

1. Show that Bianchi identities $(D_a R^d_{ebc} + D_c R^d_{eab} + D_b R^d_{eca})$ imply that the Einstein tensor, $G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R$, has vanishing divergence, $D_a G^a_b = 0$.

   Hint: multiply Bianchi identities by two Kronneker delta's with carefully chosen indexes.

2. Calculate the Schwarzschild radius for the Sun ($M_\otimes \approx \ 2.0 \cdot 10^{30} \ kg$), the Earth ($M_\oplus \approx \ 6.0 \cdot 10^{24} \ kg$) and the proton ($M_p \approx \ 938 \ MeV$).

3. What is the Kepler's law for a circular orbit (that is the relation between the orbit's period and radius) in the Schwarzschild metric?

   Hints: Period=$2\pi/\omega$, where $\omega = d\phi/dt$ is the angular frequency which can be found from the geodesic equation $Du^r = 0$. Answer: like in Newtonian theory, $\omega^2 = M/r^3$.