General relativity: note8

The Newtonian limit: slow motion in weak fields

In this limit all fields are weak and all velocities are small. Only the $_{00}$ component of the energy-momentum tensor is non-vanishing, $T_{00} = \mu$ where μ is the mass density of the matter. Therefore we shall only consider the $_{00}$ component of the Einstein's equation, $R_{00} = \kappa(T_{00} - \frac{1}{2}g_{00}T)$.

In the Newtonian limit $g_{00} = 1 + 2\phi$, $\Gamma^{\alpha}_{00} = -\phi^{,\alpha}$, $R_{00} = -\phi^{;\alpha}_{,\alpha} = \Delta\phi$, where $\alpha = 1, 2, 3$. The Einstein equation thus turns into the the Poisson's equation $\Delta\phi = \frac{1}{2}\kappa\mu$ which is equivalent to the Newtonian theory if we put $\kappa = \frac{8\pi G}{c^4}$, where $G = 6.67 \cdot 10^{-11} m^3 kg^{-1} s^{-2}$ is the Newton's gravitational constant.

Schwarzschild metric: a static spherically symmetric solution of vacuum Einstein equations

A general spherically symmetric static metric can be written as

$$ds^2 = Adt^2 - Bdr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where A and B are functions of only the "radius" r. The Christoffel symbols $\Gamma^a_{bc} = \frac{1}{2}g^{ad}(g_{db,c} - g_{bc,d} + g_{cd,b})$ are given as

$$\Gamma^r_{rr} = \frac{1}{2}\frac{B'}{B},\ \Gamma^t_{tr} = \frac{1}{2}\frac{A'}{A},\ \Gamma^r_{tt} = \frac{1}{2}\frac{A'}{B},\ \Gamma^\theta_{\theta r} = \frac{1}{r},\ \Gamma^r_{\theta\theta} = -\frac{r}{B}, \tag{2}$$

$$\Gamma^\phi_{\phi r} = \frac{1}{r},\ \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{B},\ \Gamma^\phi_{\phi\theta} = \cot\theta,\ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta. \tag{3}$$

The Ricci tensor $R_{ab} = R^c_{acb}$, where $R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}$ is then equal

$$R_{tt} = \frac{A''}{2B} + \frac{A'}{B}(\frac{1}{r} - \frac{B'}{4B} - \frac{A'}{4A}),\ R_{\theta\theta} = 1 - (\frac{r}{B})' - \frac{1}{2}(\frac{A'}{A} + \frac{B'}{B})\frac{r}{B},\ R_{rr} = -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB}, \tag{4}$$

The vacuum Einstein equations are $R_{ab} = 0$. Making a linear combination $BR_{tt} + AR_{rr} = 0$ we find $A'B + AB' = 0 \Rightarrow \frac{A'}{A} + \frac{B'}{B} = 0 \Rightarrow AB = 1$. From $R_{\theta\theta} = 0$ we then find $\frac{r}{B} = r - R$, where R is an integration constant, and $B = \frac{1}{1-\frac{R}{r}}$, $A = 1 - \frac{R}{r}$. Finally the famous Schwarzschild metric is

$$ds^2 = (1 - \frac{R}{r})dt^2 - (\frac{1}{1 - \frac{R}{r}})dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5}$$

The integration constant R is determined from the Newtonian limit, $R = 2GM$, where M is the mass of the central body. It is called gravitaional radius, or Schwarzschild radius.

Exercises

1. Calculate the energy-momentum tensor T_{ab} for a particle with mass m. The action is $S = -m\int ds$.
2. For the metric $ds^2 = (1 - \frac{R}{r})dt^2 - \frac{1}{1-\frac{R}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$ write down the geodesic equations for a massive body and for a ray of light.