General relativity: note7

Hilbert action for the gravitational field

The most successful action for the gravitational field is the Hilbert action

$$S_g = -\frac{1}{2\kappa}\int R\sqrt{-g}d\Omega, \tag{1}$$

where κ is a constant (Einstein's constant). Its variation δS_g under a variation δg^{ab} is

$$\delta S_g = -\frac{1}{2\kappa}\int (R_{ab} - \frac{1}{2}g_{ab}R)\delta g^{ab}\sqrt{-g}d\Omega. \tag{2}$$

Gravitational field equations (Einstein equations)

From the least action principle $\delta S_g + \delta S_m = 0$, where the variation of the matter action δS_m is

$$\delta S_m = \frac{1}{2}\int T_{ab}\delta g^{ab}\sqrt{-g}d\Omega, \tag{3}$$

we find the gravitational field equations

$$R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab}. \tag{4}$$

Exercises

1. From the variational principle $\delta S = 0$ derive the corresponding equations of motion by directly calculating the variation of the action for the following systems (flat space unless stated otherwise):
   - Nonrelativistic particle in a potential V: $S = \int \left(\frac{1}{2}m\vec{v}^2 - V(\vec{r})\right) dt$ [answer: $m\frac{d^2\vec{r}}{dt^2} = -\frac{dV}{d\vec{r}}$]
   - Free relativistic particle in a curved space: $S_p = -m\int ds$ [answer: $\frac{du_a}{ds} = \frac{1}{2}g_{bc,a}u^b u^c$]
   - Free electromagnetic (EM) field: $S_{em} = -\frac{1}{8\pi}\int A^a_{,b}A_a^{;b}d\Omega$ [answer: $A^{a,b}_{,b} = 0$]
   - EM field created by a given current j_a: $S = -\int A^a j_a d\Omega + S_{em}$ [answer: $A^{a,b}_{,b} = 4\pi j^a$]
   - A particle with charge q in an EM field A^a: $S = -mc\int ds - q\int A^a dx_a$ [answer: $m\frac{du^a}{ds} = F^{ab}u_b$]