**General relativity: note6**

The differential $dg$ of the determinant of the metric tensor $g_{ab}$ is $dg = gg^{ab}dg_{ab} = -gg_{ab}dg^{ab}$

## Variational principle (= least action principle)

The laws of physics can be formulated as a vanishing variation, $\delta S = 0$, of some action $S = \int Ld\Omega$, where $L$ is called Lagrangian (density). Examples:

1. Nonrelativistic particle: $L = T - V$, where $T = \frac{1}{2}mv^2$ is the particle's kinetic energy and $V$ is its potential energy.
2. Free relativistic particle: $S = -mc\int ds$.
3. Electromagnetic field: $L_{em} = -\frac{1}{16\pi}F_{ab}F^{ab} \propto -\frac{1}{8\pi}A^a_{,b}A^{;b}_a$.
4. Relativistic particle with charge q in the electromagnetic field $A^a$: $= -mc\int ds - q\int A^a dx_a + L_{em}$

## The action for the matter in the presence of a gravitational field

If the action S for a physical system in special relativity is $S_m = \int Ld\Omega$, then in a gravitational field it must have basically the same form, $S_m = \int L[g]\sqrt{-g}d\Omega$, where the notation $L[g]$ indicates that all contractions in the Lagrangian L must be written explicitely through the metric tensor $g_{ab}$.

## Variation of the action with respect to $g_{ab}$ and the energy-momentum tensor of matter

The variation $\delta S_m$ of the matter action due to a variation $\delta g^{ab}$ is given in terms of a symmetric tensor $T_{ab}$,

$$\delta S_m = \frac{1}{2}\int T_{ab}\delta g^{ab}\sqrt{-g}d\Omega = -\frac{1}{2}\int T^{ab}\delta g_{ab}\sqrt{-g}d\Omega, \tag{1}$$

where

$$\frac{1}{2}\sqrt{-g}T_{ab} = \frac{\delta(\sqrt{-g}L)}{\delta g^{ab}}. \tag{2}$$

This tensor satisfies the equation $T^{ab}_{;b} = 0$ which in a flat space turns into the energy-momentum conservation equation $T^{ab}_{,b} = 0$ and we thus deduce that it must be the energy-momentum tensor.

## Exercises

1. Prove, that $\Gamma^a_{ba} = (\ln\sqrt{-g})_{,b}$ (where $_{,b} \equiv \frac{\partial}{\partial x^b}$).
2. Prove, that $\sqrt{-g}A^a_{;a} = (\sqrt{-g}A^a)_{,a}$ (where $_{;b} \equiv \frac{D}{\partial x^b}$).
3. The lagrangian density for the electromagnetic field is $L = -\frac{1}{16\pi}F_{ab}F^{ab}$. Calculate the corresponding energy-momentum tensor using $\frac{1}{2}\sqrt{-g}T_{ab} = \frac{\partial\sqrt{-g}L}{\partial g^{ab}}$. Answer: $T_{ab} = \frac{1}{4\pi}(-F_{ac}F^c_b + \frac{1}{4}F_{cd}F^{cd}g_{ab})$