General relativity: note5

The curl theorem, $\oint_{\partial S} \vec{H} \cdot d\vec{l} = \int_S rot\vec{H} \cdot d\vec{S}$, is a variant of the Stoke's theorem. The parallel transform of a vector is such, that the covariant derivative, $DA_a = dA_a - \Gamma^c_{ab} A_c dx^b$, vanishes.

The Riemann curvature tensor

If we make a parallel transform (ie, $DA^a = 0$ along the way) of a vector A^a along an infinitesimal closed contour the components of the vector will generally change (ie, $dA^a \neq 0$) if the space is curved. The change ΔA_a of the a-th component will be proportional to the vector itself and the area ΔS^{bc} of the surface enclosed by the contour

$$\Delta A_a = \frac{1}{2} R^d_{abc} A_d \Delta S^{bc}. \tag{1}$$

The factor R^d_{abc} here is called the Riemann tensor. As we have easily calculated

$$R^d_{abc} = \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab} + \Gamma^d_{eb} \Gamma^e_{ac} - \Gamma^d_{ec} \Gamma^e_{ab}. \tag{2}$$

The Riemann tensor defines also the commutator of covariant derivatives

$$(D_a D_b - D_b D_a) A_c = R^d_{abc} A_d. \tag{3}$$

Properties of the curvature tensor

(read about them in your textbook).

$$R_{abcd} = -R_{bacd},\ R_{abcd} = -R_{abdc},\ R_{abcd} = R_{cdab},\ R_{a[bcd]} = 0,\ R_{ab[cd;e]} = 0, \tag{4}$$

where the square brackets denote symmetrisation over the indices and the semi-colon is a covariant derivative. The fourth and fifth identities are sometimes called the "algebraic Bianchi identity" and the "differential Bianchi identity", respectively.

Ricci tensor

$$R_{ab} = R^d_{adb} \tag{5}$$

Ricci scalar

$$R = g^{ab} R_{ab} \tag{6}$$

Exercises

1. Compute all the non-vanishing components of the Riemann tensor $R_{abcd}(wherea, b, c, d = \theta, \phi)$ for the metric

$$ds^2 = r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{7}$$

   on a 2-dimensional sphere of radius r. Calculate also the Ricci tensor R_{ab} and the scalar curvature R.

   Answer:

$$R_{\theta\phi\theta\phi} = r^2 Sin^2\theta = R_{\phi\theta\phi\theta} = -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi} \tag{8}$$

2. In a suitable coordinate system the gravitational field of the earth is approximately (to the lowest order in $M/r \ll 1$)

$$ds^2 = (1 - 2M/r)dt^2 - (1 + 2M/r)(dx^2 + dy^2 + dz^2). \tag{9}$$

   Suppose a satellite orbits the earth in a circular equatorial orbit ($u^r = u^\theta = 0$). What is the orbital period? What is the period in the Newtonian theory?

   Hints: Period=$2\pi/\omega$, where $\omega = d\phi/dt$ is the angular frequency which can be found from the geodesic equations $Du^r = 0$.

   Answer: $\omega^2 = M/r^3$