Contravariant and covariant vectors

A set of four quantities A^a, where a=0,1,2,3, is called a **contravariant vector** if under a transformation of coordinates $x \to x'(x)$ it transforms as coordinate differentials dx^a,

$$A^a = (\partial x^a / \partial x'^b) A'^b .$$

A set of four quantities A_a is called a **covariant vector** if under a transformation of coordinates $x \to x'(x)$ it transforms as derivatives of a scalar $d\phi / dx^a$,

$$A_a = (\partial x'^b / \partial x^a) A'_b .$$

The **contraction** $A^a B_a \equiv \Sigma_a A^a B_a$ is invariant under coordinate transformation $A^a B_a = A'^a B'_a$

Covariant differentiation

Covariant differentials DA^a and DA_a in a curved space contain Christoffel symbols $\Gamma^a{}_{bc}$

$$DA^a = dA^a + \Gamma^a{}_{bc} A^b dx^c , \quad DA_a = dA^a - \Gamma^b{}_{ac} A_b dx^c$$

Basically all physical laws in curved space are the same as in flat space, only one has to chande the small differential "d" into the covariant differential "D".

Christoffel symbols and the metric tensor

The metric tensor g_{ab} defines the invariant length element in curved coordinates

$$ds^2 = g_{ab} dx^a dx^b .$$

The metric tensor also connects contra- and covariant vectors

$$A^a = g^{ab} A_b .$$

Considering $DA_a = g_{ab} DA^b = Dg_{ab} A^b$, the covariant derivative of the metric tensor is zero

$$Dg_{ab} = 0 ,$$

from where one can then find

$$\Gamma_{a,bc} = {}^1/_2 ({}^{dg_{ab}}/_{dx^c} - {}^{dg_{bc}}/_{dx^a} + {}^{dg_{ac}}/_{dx^b})$$

Geodesic as the constant velocity trajectory.

A particle in a gravitation field moves in such a way that the covariant derivative of its velosity u^a is vanishing

$$Du^a / ds = 0 ,$$

$$d^2x^a / ds^2 + \Gamma^a{}_{bc}\, dx^b / ds\; dx^c / ds = 0 .$$

This line is called *geodesic*.

Exercises

1. For the Rindler space with the metric $ds^2 = g^2 \rho^2 d\eta^2 - d\rho^2$
   a) Find g_{ab}, g^{ab} and calculate the Christoffel symbols.
   b) Using these Christoffel symbols write down the geodesic equations and compare with the equations for the motion of a free particle from note 2.
2. Consider the polar coordinates $x = r \cos\theta$, $y = r \sin\theta$ in a two-dimentional flat space.
   a) Assume that geodesics are the usual straight lines and find the geodesic equations (as in note 2).
   b) From the line element $ds^2 = dr^2 + r^2 d\theta^2$ find the metric tensor g_{ab}, g^{ab} and the Christoffel symbols and write down the geodesic equation.
