The non-inertial Rindler frame (named after Wolfgan Rindler?) is made of constantly, uniformly accelerated clocks. The accelerations are tuned such that if two of the clocks are connected with a solid rod, no stresses appear in the rod in the process of motion. In other words it is, in principle, possible to realize the Rindler frame with material bodies. The Rindler frame is very useful as a model of a constant uniform gravitational field. It is often used as a test-ground for all sorts of theories. We shall see on the example of the Rindler frame that non-inertial frames have some peculiar features: they are curved, they have horizons, time goes differently at different places.

Consider a 1+1 dimentional inertial frame K with coordinates (t,x) where X-axis is pointing up. The non-inertial frame K° is made of clocks which are constantly accelerating upwards. The clocks are connected with solid rods of unit length. The bottom clock is marked with zero and is accelerating with acceleration g. The next clock is marked with number 1, the next with number 2 and so forth, making a ruler to measure distances. Let us designate the time and distance in this system as (τ,ξ).

The relativistic equation of motion of a body accelerating with a constant acceleration *a*:

$$^{d}/_{dt}\left(^{v}/_{\sqrt{[1-v^2]}}\right) = a, \rightarrow {}^{v}/_{\sqrt{[1-v^2]}} = at, \rightarrow v = {}^{at}/_{\sqrt{[1+(at)^2]}}, \rightarrow x = \mathrm{const}+{}^{1}/_{a}\sqrt{[1+a^2t^2]}.$$

The clock at distance ξ accelerate with acceleration $a(\xi)$. Since there are no stresses in the connecting rods the acceleration must satisfy the equation $da = -a^2 d\xi$. The solution is ${}^{1}/_{a} = \xi+{}^{1}/_{g}$, where we have used that $a(\xi=0) = g$.

Now in the inertial frame the eqution of motion of a clock marked ξ with acceleration $a(\xi) = (\xi+{}^{1}/_{g})^{-1}$ is $x = (\xi+{}^{1}/_{g})\sqrt{[1+t^2/(\xi+{}^{1}/_{g})^2]}$.

We can choose the time τ to be the proper-time of the clocks. Thus the relativistic invariant $ds^2 = dt^2-dx^2$ for a constant ξ should be equal $ds^2|_{d\xi=0} = d\tau^2$. This immediately leads to the following transformation rules:

$$x = (\xi+1/g)\cosh({}^{\tau}/_{(\xi+1/g)})-1/g\ ,\ t = (\xi+1/g)\sinh({}^{\tau}/_{(\xi+1/g)})\ ,\ ds^2 = d\tau^2-\left(1+{}^{\tau^2}/_{(\xi+1/g)^2}\right)d\xi^2.$$

The system is apparently curved in a wierd sort of way. Let's streighten it up a bit with new coordinates $g\eta={}^{\tau}/_{(\xi+1/g)}$ and $\rho=\xi+1/g$:

$$x = \rho\cosh(g\eta)-1/g\ ,\ t = \rho\sinh(g\eta)\ ,\ ds^2 = (g\rho)^2 d\eta^2 - d\rho^2\ .$$

We notice important effects in the accelerated frame:

- The Rindler coordinates are *curved*: $ds^2 = dt^2 - dx^2 = (g\rho)^2 d\eta^2 - d\rho^2$
- The local clock rate in the Rindler frame varies with height: $ds^2|_{d\rho=0} = (g\rho)^2 d\eta^2$
- The lines η=constant converge towards the *horizons* -- the boundaries that separate the part of the total space-time unavailable for the observer in the elevator.

The equivalence principle tells that all these effects must as well be present in gravitational fields.

Exercises

1. Within the inertial frame {t, x} the trajectory x(t) of a free moving body satisfies the equation ${}^{d^2x}/_{dt^2}=0$. What are the corresponding equations for the free moving body in the non-inertial frame $\{\eta, \rho\}$?
   a) Instead of x(t) consider a parametric representation of the trajectory, $\{t(\lambda), x(\lambda)\}$, with the corresponding system of equations ${}^{d^2x}/_{d\lambda^2}=0$, ${}^{d^2t}/_{d\lambda^2}=0$, where λ is some parameter.
   b) Perform a substitution of variables $x = \rho\cosh(g\eta)-1/g$, $t = \rho\sinh(g\eta)$.
   c) Make suitable linear combinations of the equations to reduce them to the following canonical geodesic form:

$$^{d^2\eta}/_{d\lambda^2} + {}^{2}/_{\rho}\,{}^{d\eta}/_{d\lambda}\,{}^{d\rho}/_{d\lambda} = 0\ ,\ \ {}^{d^2\rho}/_{d\lambda^2} + g^2\rho\,{}^{d\eta}/_{d\lambda}\,{}^{d\eta}/_{d\lambda} = 0\ .$$

2. Estimate the difference between the clock rates at the bottom and at the top of the mount Everest in the Great Himalayas.
