General relativity: note14

Space-time of an isotropic universe. Cosmological redshift. Hubble constant.

The Friedman's equation for the isotropic universe, $\frac{3}{a^4}(a^2+a'^2)=\kappa\epsilon$, and the energy conservation equation, $d\epsilon=-(\epsilon+p)3\frac{da}{a}$, can be integrated for the matter dominated universe, where the pressure is zero, $p=0$, and the energy density ϵ is equal to the mass density μ, $\epsilon=\mu$. The energy conservation gives $\mu a^3 = Const$ and the subsequent integration of the Friedmann's equation gives for a closed universe ($a^2>0$) a "Big Bang"→"Big Crunch" scenario:

$$a=a_0(1-\cos(\eta))\ ,\ t=a_0(\eta-\sin(\eta)) \tag{1}$$

For the open isotropic universe the Friedmann's equation provides a "Big Bang"→"Expansion Forever" scenario:

$$a=a_0(\cosh(\eta)-1)\ ,\ t=a_0(\sinh(\eta)-\eta) \tag{2}$$

For a flat isotropic universe $ds^2=dt^2-b^2(t)(dx^2+dy^2+dz^2)$ the scenario is also "Big Bang"→"Expansion Forever" (see the Exercise): $\mu b^3=const, b=const\ t^{2/3}$.

At early stages with high densities the universe was (probably) rather radiation dominated, $p=\frac{\epsilon}{3}$. This, however, doesn't save us from the singular point at $\eta=0$. Indeed, we have (for $\eta\ll 1$) : $\epsilon a^4=const\ ,\ a=const\cdot t^{1/2}$

Cosmological redshift. Hubble constant.

In an isotropic universe the radial ($d\theta=d\phi=0$) propagation of light ($ds^2=0$) is described by $\chi=\pm\eta+const$, from where one can deduce that along the light ray there remains a constant product $\omega a=const$. A light ray with frequency ω_0 emitted at a distance χ and observed at the origin ($\chi=0$) at time η should then have the frequency $\omega=\omega_0\frac{a(\eta-\chi)}{a(\eta)}\approx\omega_0(1-\chi\frac{a'}{a})$, that is, redshifted, if the universe expands ($a'>0$). The proper distance l to the source of light is $l=\chi a$. Thus the frequency shift z can be written as $z\equiv\frac{\omega_0-\omega}{\omega_0}=\frac{a'}{a^2}l\equiv Hl$, where H is the so called Hubble constant, $H=\frac{a'}{a^2}=\frac{1}{a}\frac{da}{dt}$. The current value of the Hubble constant is $H\approx\frac{1}{(13bil.years)}$.

Inserting $\frac{a'}{a^2}=H$ into Friedmann's equations leads to $\frac{1}{a^2}=H^2-\frac{\kappa\mu}{3}$ for a closed universe, and to $\frac{1}{a^2}=\frac{\kappa\mu}{3}-H^2$ for an open universe. For the critical density μ_c, such that $\frac{\kappa\mu_c}{3}=H^2$, the universe is flat.

The current measurements show that the relative density $\Omega=\frac{\mu}{\mu_c}$ is close to one with an error about few per cent (flatness problem). About 30% of it is "dark matter" and about 70% is "dark energy". The visible matter constitutes only about 3% of the density.

Exercises

1. Consider a flat (Euclidean) isotropic universe with the metric $ds^2=dt^2-b^2(t)(dx^2+dy^2+dz^2)$:
   (a) Calculate the Christoffel symbols. $[\Gamma^x_{tx}=\Gamma^y_{ty}=\Gamma^z_{tz}=\frac{b'}{b},\ \Gamma^t_{xx}=\Gamma^t_{yy}=\Gamma^t_{zz}=bb']$
   (b) Calculate the Ricci tensor and the Ricci scalar. $[R^t_t=-3\frac{b''}{b},\ R^x_x=R^y_y=R^z_z=-\frac{b''}{b}-2\frac{b'^2}{b^2}]$
   (c) Write down the ^t_t component of the Einstein's equations (with perfect fluid). $[3\frac{b'^2}{b^2}=\kappa\epsilon]$
   (d) Write down the energy conservation equation $\frac{dV}{V}=-\frac{d\epsilon}{(\epsilon+p)}$. $[3\ln(b)=-\int\frac{d\epsilon}{(\epsilon+p)}]$
   (e) Integrate the equations for a matter dominated universe ($p=0,\ \epsilon=\mu$). $[\mu b^3=const, b=const\cdot t^{2/3}]$
   (f) Integrate the equations for a radiation dominated universe ($p=\frac{\epsilon}{3}$). $[\epsilon b^4=const, b=const\cdot t^{1/2}]$
2. Interpret the cosmological red shift $\frac{\omega_0-\omega}{\omega_0}=Hl$ (l is the distance to the "red shifted" galaxy) as a Doppler effect and calculate the velocity with which a galaxy appears to be moving relative to the observer.