General relativity: note13

Geometry of a homogeneous isotropic universe. Friedman metric.

Asume that the universe has always been homogeneous and isotropic (Friedman model). The metric can then be chosen as $ds^2 = dt^2 - dl^2$, where the isotropic three-dimensional line element dl^2 can be written as

$$dl^2 = B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{1}$$

The Ricci tensor (borrowed from the Schwarzschild metric calculation) is then equal

$$R_{rr} = \frac{B'}{rB} \ , \ R_{\theta\theta} = 1 - \frac{1}{B} + \frac{rB'}{2B^2} \ . \tag{2}$$

In the isotropic space (= space with constant curvature) the Ricci tensor must have the form $R_{ab} = 2\lambda g_{ab}$, where λ is some constant. This gives $B = \frac{1}{1-\lambda r^2}$. Lambda can be negative (open universe), positive (closed universe), or equal to zero (flat universe). For the closed universe denoting $\lambda = \frac{1}{a^2} > 0$ and making a substitution $r = a\sin\chi$, where $0 \le \chi \le \pi$, gives the metric of a four-dimentional sphere

$$dl^2 = a^2\left(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right) \ . \tag{3}$$

For the closed universe denoting $\lambda = -\frac{1}{a^2} < 0$ and making a substitution $r = a\sinh\chi$, where $0 \le \chi \le \infty$, gives the metric of a four-dimentional hyperboloid

$$dl^2 = a^2\left(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right) \ . \tag{4}$$

This type of metric is called Friedman metric (or Freidman-Lemaitre-Robertson-Walker metric).

Friedman equation.

Friedman metric describes a homogeneous and isotropic universe. For the closed universe the interval is

$$ds^2 = a^2(d\eta^2 - d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)), \tag{5}$$

where $r = a\sin\chi$, η is the scaled time coordinate, $dt = ad\eta$, and $a(\eta)$ is the scale parameter of the universe (the radius of the 4-sphere). The components of the (diagonal) Ricci tensor are

$$R^\eta_\eta = (\frac{3}{a^4})(a'^2 - aa''), \ \ R^\chi_\chi = R^\theta_\theta = R^\phi_\phi = -(\frac{1}{a^4})(2a^2 + a'^2 + aa''), \ \ R = -(\frac{6}{a^3})(a + a'') \tag{6}$$

where prime denotes the derivative with respect to η.

Assuming that the universe is filled with a perfect fluid the energy-momentum tensor of the matter is $T_{ab} = (\epsilon + p)u_a u_b - pg_{ab}$ where ϵ is the energy density and p is the pressure. In our frame, where the matter is at rest, the 4-velosity $u^a = (\frac{1}{a}, 0, 0, 0)$.

The Einstein's equations $R^a_b - \frac{1}{2}R\delta^a_b = \kappa T^a_b$ will then have the ${}^\eta_\eta$ component

$$(\frac{3}{a^4})(a^2 + a'^2) = \kappa\epsilon \ , \tag{7}$$

also called Friedman's equation, and the three identical spatial equations $(\frac{1}{a^4})(a^2 + 2aa'' - a'^2) = -\kappa p$.

Exercises

1. Calculate the Ricci tensor and the Ricci scalar for the metric

$$ds^2 = a^2(\eta)\left(d\eta^2 - d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{8}$$

2. Calculate the volume of the closed and open universes.
3. Prove that for a perfect fluid, $T_{ab} = (\epsilon + p)u_a u_b - pg_{ab}$, no solution of the Einstein's equation is homogenius, isotropic and static. Hints: $\epsilon > 0$, $p > 0$, $T^a_a \ge 0$.
4. Prove, that with a cosmological constant, $R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab} + \Lambda g_{ab}$, a static solution does exist.