## General relativity: note11

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### Mercury perihelion advance.

In the 19th century it was discovered that interplantary perturbations cannot account fully for the turning rate of the Mercury's orbit. About 43 arcseconds per century remained unexplained. The general theory of relativity exactly accounts for this descrepancy.

The newtonian equation for the trajectory of a planet, $u'' + u = \frac{M}{J^2}$, (where $u = 1/r$ and $u' = \frac{du}{d\phi}$) has a periodic elliptic solution $u = A\sin(\phi - \phi_0) + \frac{M}{J^2}$ with an angular period $T = 2\pi$. The corresponding relativistic equation,

$$u'' + u = \frac{M}{J^2} + 3Mu^2 \ , \tag{1}$$

has an additional relativistic term $3Mu^2$ which causes the perihelion to shift.

Let us try to find a correction $\epsilon$ to the angular frequency by searching for a solution in the form $u = A\sin(1+\epsilon)\phi + B$. Setting this into the equation gives

$$-A2\epsilon\sin(1+\epsilon)\phi = 3MA2B\sin(1+\epsilon)\phi \ , \tag{2}$$

which gives $\epsilon = -\frac{3M^2}{J^2}$ and correspondingly the shift of the orbit, $\Delta\phi = 2\pi\frac{3M^2}{J^2}$. This accounts precisely for the unexplained advance of the orbit.

### Bending of light.

General relativity predicts apparent bending of light rays passing through gravitational fields. The bending was first observed in 1919 by Sir Arthur Stanley Eddington during a total eclipse when stellar images near the occulted disk of the Sun appeared displaced by some arcseconds from their usual locations in the sky. Again, extended massive objects such as galaxies may act as gravitational lenses, providing more than one optical path for light emanating from a source far behind the lens and thus producing multiple images. Such multiple images, typically of quasars, had been discovered by the early 1980s.

In the newtonian theory the light rays travel along straight lines described by and equation $u'' + u = 0$ with the straight-line solution $u = A\sin(\phi - \phi_0)$. The corresponding relativistic equation

$$u'' + u = 3Mu^2 \tag{3}$$

has an additional term, which causes the light trajectory to deflect from the straight line. Searching for the solution in the form $u = A\cos\phi + \epsilon$, where $\epsilon$ is a small correction, gives $\epsilon = MA^2(2 - \cos^2\phi)$. This gives the angle of deflection between the in-going and out-going rays $\Delta\phi = 4MA = \frac{4M}{r_0}$ where $r_0$ is the closest distance between the ray and the central body.

### Exercises

1. Show that a light ray can travel around a massive star in a circular orbit much like a planet. Calculate the radius (in Schwarzschild coordinates) of this orbit. (Answer: $r = \frac{3}{2}r_g$)

2. Calculate the deflection angle of a non-relativistic particle in a hyperbolic orbit around the star (and compare with the computed deflection of the light ray).