General relativity: note10

Motion in the Schwarzschild metric

In the Schwarzschild metric the geodesic equations, $\frac{d(g_{ab}u^b)}{ds} = \frac{1}{2}g_{bc,a}u^b u^c$, for $a = t, \theta, \phi$ are:

$$\frac{d}{ds}\left[\left(1-\frac{2M}{r}\right)\frac{dt}{ds}\right] = 0 \; ; \; \frac{d}{ds}\left[r^2\frac{d\theta}{ds}\right] = r^2 \sin\theta\cos\theta(\frac{d\phi}{ds})^2 \; ; \; \frac{d}{ds}\left[r^2\sin^2\theta\frac{d\phi}{ds}\right] = 0 \; . \tag{1}$$

Instead of the $a = r$ geodesic we shall divide the expression for the Schwarzschild metric by ds^2:

$$1 = \left(1-\frac{2M}{r}\right)(\frac{dt}{ds})^2 - \left(1-\frac{2M}{r}\right)^{-1}(\frac{dr}{ds})^2 - r^2\left[(\frac{d\theta}{ds})^2 + \sin^2\theta(\frac{d\phi}{ds})^2\right] \tag{2}$$

The first three equations can be integrated as $\theta = \pi/2$, $r^2\frac{d\phi}{ds} = J$, $(1-\frac{2M}{r})\frac{dt}{ds} = E$, where J and E are constants. The fourth equation then bocomes

$$1 = \left(1-\frac{2M}{r}\right)^{-1}E^2 - \left(1-\frac{2M}{r}\right)^{-1}J^2r'^2 - \frac{J^2}{r^2} \; , \tag{3}$$

where $r' \equiv \frac{dr}{d\phi}$. Traditionally one makes a variable substitution $r = 1/u$

$$(1-\frac{2M}{r})u = E^2 - J^2u'^2 - J^2u^2(1-2Mu) \tag{4}$$

which is the sought equation of motion.

Differentiating it once more and assuming $u' \neq 0$ gives

$$u'' + u = \frac{M}{J^2} + 3Mu^2 \tag{5}$$

Exercises

1. Calculate $R_{\theta\theta}$ and R_{rr} from note8.
2. Show that in a synchronous reference system ($ds^2 = d\tau^2 + g_{\alpha\beta}dx^\alpha dx^\beta$, where $\alpha, \beta = 1, 2, 3$) the time lines are geodesics.
3. Show that a light ray can travel around a massive star in a circular orbit much like a planet. Calculate the radius (in Schwarzschild coordinates) of this orbit. (Answer: $r = \frac{3}{2}r_g$)
4. Find explicitely the Schwarzschild coordinates t, r as function of the Lemaitre coordinates τ, ρ. Answer: $\rho - \tau = \frac{2}{3}\frac{r^{3/2}}{r_g^{1/2}}$.