Canonical quantization of spin-$\frac{1}{2}$ field

Euler-Lagrange equation, current, energy

The simplest covariant Lagrangian for a spin-$\frac{1}{2}$ field is given as[1]

$$\mathcal{L} = i\bar{\psi}\gamma^a\partial_a\psi - m\bar{\psi}\psi \ . \tag{1}$$

The corresponding Euler-Lagrange equation,

$$(i\gamma^a\partial_a - m)\psi = 0 \ , \tag{2}$$

is called the *Dirac equation*.

The Noether's conserved current is

$$j^a = i\left(\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial_a\bar{\psi})} - \frac{\partial\mathcal{L}}{\partial(\partial_a\psi)}\psi\right) = \bar{\psi}\gamma^a\psi \ . \tag{3}$$

And the canonical energy density is[2]

$$T_0^0 = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\partial_0\psi + \partial_0\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial_0\bar{\psi})} - \mathcal{L} = -i\bar{\psi}\vec{\gamma}\vec{\partial}\psi + m\bar{\psi}\psi \ . \tag{4}$$

Normal modes

Let us find the complete basis of solutions to the Dirac equation in the form of plane waves,

$$\psi(x) = \begin{pmatrix}\phi\\ \chi\end{pmatrix} e^{-ipx} \ , \tag{5}$$

where $e^{-ipx} \equiv e^{i\vec{p}\vec{r}-iEt}$, and where the spinors ϕ and χ are no longer functions of time and coordinates. Substituting the plane-wave ansatz (5) into the Dirac equation (2) gives, in the Dirac basis,

$$\begin{bmatrix} E-m & -\vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & -(E+m)\end{bmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = 0 \ . \tag{6}$$

This homogeneous system of linear algebraic equations has a non-trivial solution only when the determinant of the matrix is zero: this gives[3] the usual relativistic relation between E and $\vec{p}$,

$$E^2 = m^2 + \vec{p}^{\,2} \ . \tag{7}$$

Again there are both positive- and negative-frequency solutions

$$u_{\vec{p}}\, e^{-ipx} \ , \text{ and } \ v_{\vec{p}}\, e^{+ipx} \ , \tag{8}$$

where $p^a = \{E_{\vec{p}}, \vec{p}\}$, $\vec{p}$ is an arbitrary wave-vector, and $E_{\vec{p}} = +\sqrt{m^2+\vec{p}^{\,2}}$ is the positive square root.

[1] the real part of
[2] the real part of
[3] $(\vec{\sigma}\vec{p})^2 = \vec{p}^2$

From the Dirac equation, the positive- and negative-frequency bispinors $u_{\vec{p}}$ and $v_{\vec{p}}$ are

$$u_{\vec{p}} = \begin{pmatrix}\phi \\ \frac{\vec{\sigma}\vec{p}}{E_{\vec{p}}+m}\phi\end{pmatrix} , \quad v_{\vec{p}} = \begin{pmatrix}\frac{\vec{\sigma}\vec{p}}{E_{\vec{p}}+m}\chi \\ \chi\end{pmatrix} . \tag{9}$$

where ϕ and χ are arbitrary spinors.

In the non-relativistic limit the Dirac bi-spinors turn into spinors,

$$\lim_{\vec{p}^2\ll m^2} u_{\vec{p}} = \phi \, , \ \lim_{\vec{p}^2\ll m^2} v_{\vec{p}} = \chi \, . \tag{10}$$

That's why non-relativistic spin-$\frac{1}{2}$ particles are described by spinors.

There are two linearly independent spinors, which can be chosen for the complete basis. For example, one can choose eigen-spinors of the $\frac{1}{2}\sigma_3$ operator in the frame where $\vec{p}$=0,

$$\frac{1}{2}\sigma_3\phi_\lambda = \lambda\phi_\lambda \ , \tag{11}$$

and similarly for the spinor χ.

Charge and normalization The charge density for the spin-$\frac{1}{2}$ field is

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi \ . \tag{12}$$

The charge $Q = \int_{V=1} d^3x j^0$ of the normal modes is then given as

$$Q\left(u_{\vec{p}\lambda}e^{-ipx}\right) = u^\dagger_{\vec{p}\lambda}u_{\vec{p}\lambda} = \phi^\dagger_\lambda\phi_\lambda\frac{2E_{\vec{p}}}{E_{\vec{p}}+m} \ , \tag{13}$$

$$Q\left(v_{\vec{p}\lambda}e^{+ipx}\right) = v^\dagger_{\vec{p}\lambda}v_{\vec{p}\lambda} = \chi^\dagger_\lambda\chi_\lambda\frac{2E_{\vec{p}}}{E_{\vec{p}}+m} \ . \tag{14}$$

Unlike the scalar field, the spin-$\frac{1}{2}$ field seemingly gives positive charges for both positive- and negative-frequency solutions.

Unit charge condition for a plane-wave leads to the normalizations

$$u^\dagger_\lambda u_{\lambda'} = v^\dagger_\lambda v_{\lambda'} = \delta_{\lambda\lambda'} \ , \tag{15}$$

which are obtained by choosing

$$\phi^\dagger_\lambda\phi_{\lambda'} = \chi^\dagger_\lambda\chi_{\lambda'} = \frac{E_{\vec{p}}+m}{2E_{\vec{p}}}\delta_{\lambda\lambda'} \ . \tag{16}$$

With this normalization

$$\bar{u}_\lambda u_{\lambda'} = -\bar{v}_\lambda v_{\lambda'} = \frac{m}{E_{\vec{p}}}\delta_{\lambda\lambda'} \ . \tag{17}$$

Energy From (4) it follows that the positive-energy solution has positive energy,

$$E\left(ue^{-ipx}\right) = \bar{u}\vec{\gamma}\vec{p}u + m\bar{u}u = E_{\vec{p}} \ , \tag{18}$$

while the negative-energy solution seemingly has negative energy,

$$E\left(ve^{+ipx}\right) = \bar{v}\vec{\gamma}(-\vec{p})v + m\bar{v}v = -E_{\vec{p}} \ . \tag{19}$$

Normal mode representation

An arbitrary solution to the Dirac equation can be represented as a linear combination of normal modes,

$$\psi = \sum_{\vec{p}\lambda} \left(a_{\vec{p}\lambda} u_{\vec{p}\lambda} e^{-ipx} + b^\dagger_{\vec{p}\lambda} v_{\vec{p}\lambda} e^{+ipx} \right) . \qquad (20)$$

The charge of the field is then given as

$$Q = \sum_{\vec{p}\lambda} \left(a^\dagger_{\vec{p}\lambda} a_{\vec{p}\lambda} + b_{\vec{p}\lambda} b^\dagger_{\vec{p}\lambda} \right) , \qquad (21)$$

and the Hamiltonian,

$$H = \sum_{\vec{p}\lambda} E_{\vec{p}} \left(a^\dagger_{\vec{p}\lambda} a_{\vec{p}\lambda} - b_{\vec{p}\lambda} b^\dagger_{\vec{p}\lambda} \right) . \qquad (22)$$

Generation/annihilation operators

The only way to make sense of the charge and Hamiltonian is to postulate *anti-commutation* relation for the spin-$\frac{1}{2}$ operators,

$$a_{\vec{p}\lambda} a^\dagger_{\vec{p}\,'\lambda'} + a^\dagger_{\vec{p}\,'\lambda'} a_{\vec{p}\lambda} = \delta_{\vec{p}\vec{p}\,'} \delta_{\lambda\lambda'} , \qquad (23)$$

$$b_{\vec{p}\lambda} b^\dagger_{\vec{p}\,'\lambda'} + b^\dagger_{\vec{p}\,'\lambda'} b_{\vec{p}\lambda} = \delta_{\vec{p}\vec{p}\,'} \delta_{\lambda\lambda'} . \qquad (24)$$

With these postulates the charge and the energy take the form as prescribed by the experiment,

$$Q = \sum_{\vec{p}\lambda} \left(n_{\vec{p}\lambda} - \bar{n}_{\vec{p}\lambda} \right) , \qquad (25)$$

$$E = \sum_{\vec{p}\lambda} E_{\vec{p}} \left(n_{\vec{p}\lambda} + \bar{n}_{\vec{p}\lambda} \right) , \qquad (26)$$

where $n_{\vec{p}\lambda} = a^\dagger_{\vec{p}\,'\lambda'} a_{\vec{p}\lambda}$ and $\bar{n}_{\vec{p}\lambda} = b^\dagger_{\vec{p}\,'\lambda'} b_{\vec{p}\lambda}$ are the number-of-particle and number-of-anti-particle operators with eigenvalues 0 and 1.

In canonical quantum field theory spin-$\frac{1}{2}$ particles are necessarily fermions.