Spin-$\frac{1}{2}$ field

Bispinors (Dirac spinors)

The lowest irreducible representation of the full Lorentz group $O(1,3)$, containing spin-$\frac{1}{2}$, is

$$(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\,. \quad (1)$$

The matrices from this representation transform four-component objects called *bispinors* or Dirac spinors,

$$\psi=\begin{pmatrix}\phi\\ \chi\end{pmatrix}\,, \quad (2)$$

where the two-component objects $\phi\in(\frac{1}{2},0)$ and $\chi\in(0,\frac{1}{2})$ are the ("right" and "left") *Weyl spinors*. Their infinitesimal transformation matrices are

$$\mathsf{t}_{(\frac{1}{2},0)}=1+i\tfrac{1}{2}\vec{\sigma}d\vec{w}\,, \quad (3)$$

and

$$\mathsf{t}_{(0,\frac{1}{2})}=1+i\tfrac{1}{2}\vec{\sigma}d\vec{w}^*\,. \quad (4)$$

Apparently,

$$\mathsf{t}_{(0,\frac{1}{2})}=\left(\mathsf{t}^\dagger_{(\frac{1}{2},0)}\right)^{-1}\,, \quad (5)$$

and therefore under a Lorentz transformation Λ

$$\phi\xrightarrow{\Lambda}\mathsf{t}\phi\,,\ \chi\xrightarrow{\Lambda}\mathsf{t}^{\dagger^{-1}}\chi\,, \quad (6)$$

where $\mathsf{t}\equiv\mathsf{t}_{(\frac{1}{2},0)}$.

Under parity transformation, $\mathcal{P}$, the Weyl spinors transform into each other,

$$\phi\xleftrightarrow{\mathcal{P}}\chi\,. \quad (7)$$

Bilinear forms of bispinors

The Lagrangian for the filed must be a real scalar, bilinear in the field. That is, it has to be built out of $\psi^\dagger\otimes\psi$ which transforms under the representation

$$\left((\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\right)^*\otimes\left((\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\right)\,. \quad (8)$$

According to the general rule this direct product reduces to a direct sum of the following representations,

$$(0,0)\,;\ (0,1)\oplus(1,0)\,;\ (\tfrac{1}{2},\tfrac{1}{2})\,, \quad (9)$$

where the first is scalar, the second – antisymmetric tensor, and the third – four-vector.

Scalars

Using the transformation laws (6) and (7) it is easy to show that the (real) scalar is the combination

$$\bar{\psi}\psi\equiv\phi^\dagger\chi+\chi^\dagger\phi\,, \quad (10)$$

where $\bar{\psi}\equiv\psi^\dagger\gamma^0$ and the block-matrix γ^0 is given as

$$\gamma^0=\begin{bmatrix}0 & \mathbf{1}\\ \mathbf{1} & 0\end{bmatrix}\,. \quad (11)$$

The combination

$$i\bar{\psi}\gamma^5\psi\equiv-i(\phi^\dagger\chi-\chi^\dagger\phi)\,, \quad (12)$$

where

$$\gamma^5=\begin{bmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{bmatrix} \quad (13)$$

is a (real) *pseudo-scalar*, that is, it changes sign under parity transformation.

Vectors

Since $\phi\in(\frac{1}{2},0)$ and $\phi^*\in(0,\frac{1}{2})$, their direct product is a four-vector,

$$\phi^*\otimes\phi=(\tfrac{1}{2},\tfrac{1}{2})\,. \quad (14)$$

Apparently, this four-vector is

$$\{\phi^\dagger\phi,\phi^\dagger\vec{\sigma}\phi\}\,. \quad (15)$$

Analogously,

$$\{\chi^\dagger\chi,-\chi^\dagger\vec{\sigma}\chi\} \quad (16)$$

is also a four-vector.

Now we can build a *polar vector*,

$$\bar{\psi}\gamma^a\psi\equiv\{\phi^\dagger\phi+\chi^\dagger\chi,\phi^\dagger\vec{\sigma}\phi-\chi^\dagger\vec{\sigma}\chi\}\,, \quad (17)$$

and an *axial-* or *pseudo-vector*,

$$i\bar{\psi}\gamma^a\gamma^5\psi\equiv i\{\phi^\dagger\phi-\chi^\dagger\chi,\phi^\dagger\vec{\sigma}\phi+\chi^\dagger\vec{\sigma}\chi\}\,, \quad (18)$$

where $\gamma^a=\{\gamma^0,\vec{\gamma}\}$ and

$$\vec{\gamma}=\begin{bmatrix}0 & -\vec{\sigma}\\ \vec{\sigma} & 0\end{bmatrix}\,. \quad (19)$$

Antisymmetric tensor

The representation $(0,1)$ can be made of

$$\phi^*\otimes\chi\sim(0,\tfrac{1}{2})\otimes(0,\tfrac{1}{2})\ni(0,1)\,, \quad (20)$$

and, correspondingly, $(1,0)$ out of

$$\chi^*\otimes\phi\sim(\tfrac{1}{2},0)\otimes(\tfrac{1}{2},0)\ni(1,0)\,. \quad (21)$$

Under rotations the objects $(0,1)$ and $(1,0)$ transform as pure vectors. Apparently they are

$$\phi^\dagger\vec{\sigma}\chi\in(0,1) \quad (22)$$

and

$$\chi^\dagger \vec{\sigma} \phi \in (1,0) . \quad (23)$$

Under reflections these two objects transform into each other. Therefore their sum and difference is a six-component objects that transforms through itself under the full Lorentz group $O(1,3)$.

Using *gamma*-matrices this tensor is usually written as

$$\tfrac{1}{2}\bar{\psi}(\gamma^a\gamma^b - \gamma^b\gamma^a)\psi . \quad (24)$$

Resume

The direct product $\psi^* \otimes \psi$ of two bispinors can be reduced into five irreducible covariant objects:

1. scalar, $\bar{\psi}\psi$;
2. pseudo-scalar, $i\bar{\psi}\gamma_5\psi$;
3. vector, $\bar{\psi}\gamma^a\psi$;
4. pseudo-vector, $i\bar{\psi}\gamma^a\gamma_5\psi$;
5. antisymmetric tensor, $\frac{1}{2}\bar{\psi}(\gamma^a\gamma^b - \gamma^b\gamma^a)\psi$,

where $\bar{\psi} \equiv \psi^\dagger\gamma_0$ and the block-matrices γ^a and γ^5 are called *gamma-matrices* or Dirac matrices.

Gamma matrices in Dirac basis

The bispinors and gamma-matrices in the previous section are given in the so-called Weyl or *chiral* basis. A similarity transformation

$$\begin{aligned} \psi &\rightarrow S\psi \\ \gamma^a &\rightarrow S\gamma^a S^{-1} \end{aligned} \quad (25)$$

with the matrix

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (26)$$

changes Weyl basis into Dirac or *standard* basis, where the γ-matrices are

$$\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (27)$$

$$\vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix} \quad (28)$$

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \quad (29)$$

The γ-matrices satisfy the anti-commutation relation

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2g^{ab} , \quad (30)$$

where g^{ab} is the Minkowski metric tensor. The matrix γ_5 anti-commutes with γ^a,

$$\{\gamma^a, \gamma_5\} = 0 . \quad (31)$$

Lagrangian of spin-$\frac{1}{2}$ field

The Lagrangian must be a real scalar, bilinear in ψ and $\partial_a\psi$. A suitable form is

$$\mathcal{L} = \frac{i}{2}\left(\bar{\psi}\gamma^a\partial_a\psi - \partial_a\bar{\psi}\gamma^a\psi\right) - m\bar{\psi}\psi . \quad (32)$$

Dirac equation

The general expression for the Euler-Lagrange equation,

$$\partial_a \frac{\partial\mathcal{L}}{\partial(\partial_a\bar{\psi})} = \frac{\partial\mathcal{L}}{\partial\bar{\psi}} , \quad (33)$$

with the Lagrangian (32) leads to the so-called *Dirac equation*,

$$(i\gamma^a\partial_a - m)\psi = 0 . \quad (34)$$

Energy-momentum tensor and conserved current

Again, using the general expressions, the energy-momentum tensor and the conserved current are directly obtained from the Lagrangian,

$$T_0^0 = -i\bar{\psi}\vec{\gamma}\vec{\partial}\psi + m\bar{\psi}\psi , \quad (35)$$

$$j^a = \bar{\psi}\gamma^a\psi . \quad (36)$$