Finite rotation matrix

The generators can recover not only infinitesimal, $g(d\alpha) = 1 + iId\alpha$, but (in principle) also finite elements of the group (with certain caveats). It is especially easy with *additive parameters*, where

$$g(\alpha + d\alpha) = g(\alpha)g(d\alpha) . \quad (1)$$

Indeed, in this case

$$g(\alpha + d\alpha) = g(d\alpha)g(\alpha) = (1 + iId\alpha)g(\alpha) , \quad (2)$$

and thus the group element $g(\alpha)$ satisfies the differential equation

$$\frac{\partial g(\alpha)}{\partial \alpha} = iIg(\alpha) , \quad (3)$$

with the initial condition $g(0) = 1$. One can check by direct substitution that the solution to this equation is exponential function, understood as an infinite Taylor series

$$g(\alpha) = e^{iI\alpha} \equiv \sum_{n=0}^{\infty} \frac{(iI\alpha)^n}{n!} . \quad (4)$$

For example, for the rotation around a given axis $\vec{n}$ the rotation angle θ is an additive parameter of the rotation matrix $R(\vec{n}, \theta)$,

$$\begin{aligned} R(\vec{n}, \theta + d\theta) &= R(\vec{n}, \theta)R(\vec{n}, d\theta) \\ &= (1 + i\vec{I} \cdot \vec{n}d\theta)R(\vec{n}, \theta) , \end{aligned} \quad (5)$$

and thus the rotation matrix satisfies the differential equation

$$\frac{dR}{d\theta} = i\vec{I} \cdot \vec{n}R , \quad (6)$$

with the boundary condition $R(\vec{n}, 0) = 1$. The solution is given by the Taylor series,

$$R(\vec{n}, \theta) = e^{i\vec{I}\cdot\vec{n}\theta} \equiv 1 + i\vec{I}\cdot\vec{n}\theta + \frac{(i\vec{I} \cdot \vec{n}\theta)^2}{2!} + \dots . \quad (7)$$

In practice for a finite-dimension representation there is only a finite number of terms in the series.

Direct product of two representations

In the field theory we often have to work with different types of products of covariant quantities, like $\partial_a j^a$ or $\partial_a A^b$. We need to know how such products transform.

For simplicity we shall only consider matrices, although the concepts are also defined for abstract groups.

A *direct* product $g \otimes h$ of two matrices $\{g_{ij}\}$ and $\{h_{kl}\}$ is a suitable arranged matrix made of all possible pairwise products of matrix elements of $\{g_{ij}h_{kl}\}$. If there are n^2 elements in a matrix g and m^2 elements in a matrix h, there will be $(nm)^2$ elements in a matrix $g \otimes h$.

If the matrices g and h act on column-vectors u and v of sizes n and m, then the matrices $g \otimes h$ act on a column-vector $u \otimes v$ of size nm consisting of all pairwise products of the elements of vectors u and v,

$$u \otimes v = \begin{bmatrix} u_1v_1 \\ \dots \\ u_nv_m \end{bmatrix} . \quad (8)$$

A *direct sum* $g \oplus h$ of two matrices g and h is a block-diagonal matrix made of matrices g and h,

$$g \oplus h = \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} . \quad (9)$$

The size of the direct sum is $n + m$.

The matrices $g \oplus h$ act on a column-vector $u \oplus v$ of size $n + m$ which is made of elements of vectors u and v,

$$u \oplus u = \begin{bmatrix} u \\ v \end{bmatrix} . \quad (10)$$

Suppose we have two (different) representations, $G = \{g\}$ and $H = \{h\}$, of some Lie group $\{\Lambda\}$, with the corresponding infinitesimal elements[1]

$$\begin{aligned} g &= 1_G + i\vec{I}_G\vec{\alpha} , \quad (11) \\ h &= 1_H + i\vec{I}_H\vec{\alpha} , \quad (12) \end{aligned}$$

where $\vec{\alpha}$ are the parameters of the group $\{\Lambda\}$.

It is easy to show that the generators $I_{G\otimes H}$ of a direct product $G \otimes H$ of two representations G and H of the same group is a (Kronecker) sum of the corresponding generators I_G and I_H,

$$\vec{I}_{G\otimes H} = \vec{I}_G \otimes 1_H + 1_G \otimes \vec{I}_H . \quad (13)$$

Indeed,

$$\begin{aligned} g \otimes h &= \left(1_G + i\vec{I}_G\vec{\alpha}\right) \otimes \left(1_H + i\vec{I}_H\vec{\alpha}\right) \\ &= 1_G \otimes 1_H + i\left(\vec{I}_G \otimes 1_H + 1_G \otimes \vec{I}_H\right)\vec{\alpha} \\ &= 1 + i\vec{I}_{G\otimes H}\vec{\alpha} , \quad (14) \\ &\Rightarrow \vec{I}_{G\otimes H} = \vec{I}_G \otimes 1_H + 1_G \otimes \vec{I}_H . \quad (15) \end{aligned}$$

Clebsch-Gordan theorem

The direct product $(j_1) \otimes (j_2)$ of two irreducible representations of the rotation group is a reducible representation which can be reduced into a direct sum of irreducible representations,

$$(j_1) \otimes (j_2) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \oplus(j) . \quad (16)$$

[1] $\vec{I}\vec{\alpha} \equiv \sum_k I_k\alpha_k$

Example: a direct product $\vec{a}\otimes\vec{b}$ of two vectors, $(1)\otimes(1)$, reduces to a direct sum of a scalar ($j=0$), an antisymmetric tensor ($j=1$), and a symmetric tensor with zero trace ($j=2$),

$$\vec{a}\otimes\vec{b}=\left(\vec{a}\vec{b}\right)\oplus\left(\vec{a}\times\vec{b}\right)\oplus\left(a_ib_j+a_jb_i-\frac{2}{3}(\vec{a}\vec{b})\delta_{ij}\right). \quad (17)$$

Irreducible representations of the Lorentz group

With the complex parameterization

$$d\vec{w}=\vec{n}d\theta+id\vec{v} \quad (18)$$

the infinitesimal Lorentz transformation is given as

$$\Lambda=1+i\vec{M}d\vec{w}+i\vec{N}d\vec{w}^{*}, \quad (19)$$

where the generators M and N satisfy the Lie algebra

$$\begin{aligned} M_kM_l-M_lM_k &= i\sum_m\epsilon_{klm}M_m && (20)\\ N_kN_l-N_lN_k &= i\sum_m\epsilon_{klm}N_m && (21)\\ M_kN_l-N_lM_k &= 0, && (22)\end{aligned}$$

that is, two independent rotation Lie algebras. An infinitesimal matrix t from a representation of the Lorentz group is then given as

$$t=\mathbf{1}_M\otimes\mathbf{1}_N+i\vec{M}\otimes\mathbf{1}_Nd\vec{w}+i\mathbf{1}_M\otimes\vec{N}d\vec{w}^{*}, \quad (23)$$

where $\mathbf{1}_M$ and $\mathbf{1}_N$ are the unit matrices in the spaces of M and N generators.

Thus an irreducible representation of the Lorentz group is determined by two numbers (j_M,j_N), each taking non-negative integer or half-integer values. The dimensions of the M and N-generators are then $(2j_M+1)$ and $(2j_N+1)$ correspondingly. The dimension of the representation is $(2j_M+1)(2j_N+1)$.

Direct product of two irreducible representations There exists a similar theorem for the Lorentz group,

$$(j_1,k_1)\otimes(j_2,k_2)=\sum_{j=|j_1-j_2|}^{j_1+j_2}\sum_{k=|k_1-k_2|}^{k_1+k_2}\oplus(j,k). \quad (24)$$

Rotational properties of an irreducible representation of the Lorentz group If we only consider rotations, $d\vec{v}=0$, the infinitesimal element of a Lorentz group representation (23) becomes

$$g\big|_{d\vec{v}=0}=1+i\left(\vec{M}\otimes\mathbf{1}_N+\mathbf{1}_M\otimes\vec{N}\right)\vec{n}d\theta, \quad (25)$$

which can be identified as the Kronecker sum (20) of two generators with rotation Lie algebra. A Kronecker sum of generators corresponds to a direct product of their representations.

Thus, under rotations, an irreducible representations (j_1,j_2) of the Lorentz group reduces to a direct sum of irreducible representations of the rotation group (j) with $j=|j_1-j_2|,\dots,j_1+j_2$.

Example: a four-vector $\{E,\mathbf{p}\}$ transforms under $(\frac{1}{2},\frac{1}{2})$ representation of the Lorentz group and under rotations reduces to a direct sum of a scalar E and a vector $\mathbf{p}$.

Parity transformation and irreducible representations of the Lorentz group Parity transformation is the simultaneous change of spatial coordinates,

$$P\begin{pmatrix}t\\ \vec{x}\end{pmatrix}=\begin{pmatrix}t\\ -\vec{x}\end{pmatrix}. \quad (26)$$

The parity transformation, like rotations, reflects our freedom in choosing frames of reference for a description of physical systems and therefore must be included in the group of coordinate transformations in the principle of covariance.

Under the parity transformation the rotation generators do not change, $\vec{J}\to\vec{J}$, while the velocity-boost-generators change sign, $\vec{K}\to-\vec{K}$. The "optimal" generators $\vec{M}=\frac{1}{2}(\vec{J}-i\vec{K})$ and $\vec{N}=\frac{1}{2}(\vec{J}+i\vec{K})$ transform into each other, $\vec{M}\leftrightarrow\vec{N}$.

Thus an irreducible representation of the Lorentz group (j_1,j_2) transforms under parity transformation into a representation (j_2,j_1),

$$(j_1,j_2)\overset{\mathcal{P}}{\longleftrightarrow}(j_2,j_1). \quad (27)$$

Consequently, if $j_1\neq j_2$, the representation of the covariance group of coordinate transformations has to be enlarged to the direct sum $(j_1,j_2)\oplus(j_2,j_1)$.