Exercises

1. Consider two interacting fields, a complex spin-$\frac{1}{2}$ field ψ and a real scalar field ϕ with interacting Lagrangian $\mathcal{L}_v = -g\bar{\psi}\psi\phi$.
   • Find the Euler-Lagrange equations and the Hamiltonian density T_0^0 of the system.
   • What is the natural dimension of the coupling constant?

2. Show that that the Heisenberg, Schrödinger, and interaction pictures are equivalent to the time evolution equation,

$$\frac{d}{dt}R = \frac{1}{i}[R, H]\,, \qquad (1)$$

   where R is a quantum-mechanical operator and H is the Hamiltonian of a quantum system. For simplicity assume that the operator R does no depend explicitly on time.

3. Time evolution in Heisenberg-Born-Jordan matrix mechanics:
   • Show that the Euler-Lagrange equation for a physical system is equivalent to the Hamilton equations,

$$\frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}\,,\; \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p}\,, \qquad (2)$$

   where q and p are the canonical coordinate and momentum of the system, and H is the Hamiltonian of the system.
   • Show that for any analytic function $f(q,p)$ its time derivative along the trajectories of the system is given as

$$\frac{d}{dt}f(q,p) = \{f, H\}\,, \qquad (3)$$

   where

$$\{f, H\} \equiv \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} \qquad (4)$$

   is the classical Poison bracket.
   • Show that if q and p are matrices with commutation relation $[q,p] = i$ than for any analytic functions $h(q,p)$, $f(q,p)$

$$\frac{1}{i}[f, h] = \frac{\partial f}{\partial q}\frac{\partial h}{\partial p} - \frac{\partial h}{\partial q}\frac{\partial f}{\partial p} \qquad (5)$$

   Hint: show (by induction) for powers $q^n p^m$.

4. Consider a non-interacting quantum scalar field.
   • What is the time evolution of the generation/annihilation operators?
   • What is the time evolution of the commutation realtion of the generation/annihilation operators?
   • What is the time evolution of a normal mode?