Exercises: free scalar field

1. For a scalar field prove the orthogonality of plane waves with periodic boundary condition.
2. For the Lagrangian
$$\mathcal{L} = \partial_a \phi^* \partial^a \phi - m^2 \phi^* \phi$$
find the expressions for the energy-momentum tensor, T^a_b, and the density of the conserved current, j^a.
3. Calculate the 4-momentum,
$$P^a = \int_V dV T^{0a} ,$$
and the conserved current,
$$J^a = \int_V dV j^a ,$$
for positive and negative frequency normal modes. Interpret the results.
4. Prove that different normal modes contribute to the total energy and the total charge of a scalar field independently (that is, there are no interference terms in the sums).
5. For the commutation relation,
$$aa^\dagger - a^\dagger a = 1 \, ,$$
and also for the anti-commutation relation,
$$aa^\dagger + a^\dagger a = 1 \, ,$$
find the eigenvalues of the operator $a^\dagger a$.
Hint: consider the eigenstates of the $a^\dagger a$ operator,
$$a^\dagger a |\lambda\rangle = \lambda |\lambda\rangle ,$$
and investigate the states $a|\lambda\rangle$ and $a^\dagger|\lambda\rangle$.
6. Does the charge operator Q commute with the hamiltonian operator H?
7. *Canonical commutation relation* in quantum mechanics is given as
$$[x, p_x] = i \, ,$$
where $\hbar = 1$ and $p_x = \partial\mathcal{L}/\partial\dot{x}$ is the canonical momentum (conjugate momentum).
For a field ϕ the corresponding canonical momentum is $p_\phi = \partial\mathcal{L}/\partial\dot{\phi}$.
Assuming bosonic commutation relation
$$a_{\mathbf{k}} a^\dagger_{\mathbf{k}'} - a^\dagger_{\mathbf{k}} a_{\mathbf{k}'} = \delta_{\mathbf{k}\mathbf{k}'}$$
argue that for our scalar field ϕ the canonical equal time commutation relation is given as
$$[\phi(t, \mathbf{r}), p_\phi(t, \mathbf{r}')] = i\delta(\mathbf{r} - \mathbf{r}') \, .$$