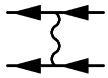
Exercises

(Extra) One-boson exchange potential

Calculate the non-relativistic *one boson exchange potential* corresponding to the diagram



(1)

for interaction Lagrangian

$$\mathcal{L}_v = -g\bar{\psi}\gamma_5\psi\phi \,, \tag{2}$$

which describes interaction of fermions with pseudo-scalar bosons.

Hints:
The matrix element[1] M is given as

$$M = i^2 g^2 \bar{u}(p_1')\gamma_5 u(p_1)\frac{1}{k^2-\mu^2}\bar{u}(p_2')\gamma_5 u(p_2) \tag{3}$$

The Dirac bispinor $u_{\mathbf{p}}$ is

$$u_{\mathbf{p}} = \begin{pmatrix} \phi_{\mathbf{p}} \\ \frac{\vec{\sigma}\mathbf{p}}{E_{\mathbf{p}}+m}\phi_{\mathbf{p}} \end{pmatrix} \tag{4}$$

which gives

$$\bar{u}(p_1')\gamma_5 u(p_1) = i\phi^\dagger_{\mathbf{p}_1'}\frac{\vec{\sigma}\mathbf{p}_1}{E_{\mathbf{p}_1}+m}\phi_{\mathbf{p}_1} \tag{5}$$

$$-i\phi^\dagger_{\mathbf{p}_1'}\frac{\vec{\sigma}\mathbf{p}_1'}{E_{\mathbf{p}_1'}+m}\phi_{\mathbf{p}_1} \tag{6}$$

In the non-relativistic limit $E \approx m$ up to the terms v^2

$$\bar{u}(p_1')\gamma_5 u(p_1) \approx \frac{i}{2m}\phi^\dagger_{\mathbf{p}_1'}\vec{\sigma}_1(\mathbf{p}_1-\mathbf{p}_1')\phi_{\mathbf{p}_1} \tag{7}$$

Now introducing $\mathbf{p}_1 - \mathbf{p}_1' = -\mathbf{q}$ gives

$$\bar{u}(p_1')\gamma_5 u(p_1) \approx -\frac{i}{2m}\phi^\dagger_{\mathbf{p}_1'}\vec{\sigma}_1\mathbf{q}\phi_{\mathbf{p}_1} \tag{8}$$

$$\bar{u}(p_2')\gamma_5 u(p_2) \approx \frac{i}{2m}\phi^\dagger_{\mathbf{p}_1'}\vec{\sigma}_2\mathbf{q}\phi_{\mathbf{p}_1} \tag{9}$$

In the c.m. frame

$$k^2 - \mu^2 = -(\mathbf{q}^2+\mu^2) \tag{10}$$

and finally

$$M = \frac{g^2}{(2m)^2}\phi_1^\dagger\phi_2^\dagger\frac{(\vec{\sigma}_1\vec{q})(\vec{\sigma}_2\vec{q})}{\vec{q}^2+\mu^2}\phi_1\phi_2 \tag{11}$$

[1] defined through

$$\langle f|S|i\rangle = -i(2\pi)^4\delta(P_f-P_i)M \,,$$

where P_i and P_f are the sums of all particle momenta in the correspindingly initial and final states.

The one-pseudo-scalar-boson-exchange-potential is the given as

$$V_{\text{PS}}(\mathbf{r}) = -\frac{g^2}{2m^2}\sigma_{1a}\sigma_{2b}\int\frac{d^3q}{(2\pi)^3}\frac{q_a q_b}{\mathbf{q}^2+\mu^2}e^{i\mathbf{qr}} \tag{12}$$

The integral

$$\int\frac{d^3q}{(2\pi)^3}\frac{q_a q_b}{\mathbf{q}^2+\mu^2}e^{i\mathbf{qr}} = \tag{13}$$

$$-\frac{\partial}{\partial r_a}\frac{\partial}{\partial r_b}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\mathbf{q}^2+\mu^2}e^{i\mathbf{qr}} = \tag{14}$$

$$-\frac{\partial}{\partial r_a}\frac{\partial}{\partial r_b}\frac{1}{4\pi}\frac{e^{-\mu r}}{r} = \tag{15}$$

$$\frac{\mu^2}{4\pi}\frac{e^{-\mu r}}{r}\left(\frac{1}{\mu r}+\frac{1}{(\mu r)^2}\right)\delta_{ab} \tag{16}$$

$$-\frac{\mu^2}{4\pi}\frac{e^{-\mu r}}{r}\left(1+\frac{3}{\mu r}+\frac{3}{(\mu r)^2}\right)\frac{r_a r_b}{r^2} \tag{17}$$

The OBEP with pseudo-scalar boson is thus a finite-range spin-spin and tensor potential of Yukawa type with the range equal to inverse mass of the exchange boson.

The OBEP with a vector boson has a slightly different spin structure which in addition includes central and spin-orbit forces. The central force has the Yukawa form $e^{-\mu r}/r$. In the limit of massless vector boson this gives the Coulomb central potential.