Exercises

1. Prove that generators of a Lie group have a Lie algebra
$$[I_l, I_m] \doteq I_l I_m - I_m I_l = i\sum_k C^k_{lm} I_k \ ,$$
where C^k_{lm} are certain numbers (called structure constants).
Hints:
• Consider the equation,
$$g(\beta') = g(\alpha)g(\beta)g^{-1}(\alpha) \,,$$
which defines β' as function of α, β. Argue that in the limit $\beta \to 0$
$$\beta'_k \approx \sum_l f_{kl}(\alpha)\beta_l \,,$$
where $f_{kl}(\alpha)$ are some functions.
• Argue that in the limit $\alpha \to 0$
$$f_{kl}(\alpha) \approx \delta_{kl} + \sum_m C^k_{lm}\alpha_m \ ,$$
where C^k_{lm} are some constants.
• Now consider the infinitesimal form of the above equation.

2. Argue that a representation of a Lie group is also a Lie group with the same Lie algebra, as the original group.

3. Show that if the operators
$$U = \exp\left(-i\sum_k J_k\alpha_k\right)$$
are unitary, $U^\dagger U = 1$, and the parameters α_k are real, then the generators J_k are hermitian: $J^\dagger_k = J_k$. Show that if $\det(U) = 1$ then $\text{trace}(J_k) = 0$.
Hints:
• consider only infinitesimal operators;

• show that $\det(e^A) = e^{\text{trace}(A)}$ by diagonalizing matrix A.

4. Using the Lie algebra of the rotation group,
$$[I_j, I_k] = i\sum_l \epsilon_{jkl} I_l \,,$$
where ϵ_{jkl} is the Levi-Civita (absolutely antisymmetric) symbol, find the 2x2 representation of the rotation generators $\vec{I}$ (assuming I_3 is diagonal and I_1 is real).
Hints:
• recall that $\mathsf{trace}(I_k) = 0$ and $I^+_k = I_k$;
• I's must be half the Pauli's σ-matrices.

5. Argue that the canonical commutation relation (for one dimensional motion, for simplicity),
$$\frac{1}{i}[x, p] = 1 \,,$$
corresponds to the classical Poisson bracket,
$$\{f(x), p\} \doteq \frac{\partial f}{\partial x} \,.$$
Hints:
• Prove by induction that for any analytic function $f(x) = \sum_n a_n x^n$
$$\frac{1}{i}[f(x), p] = \frac{\partial f}{\partial x} \,.$$