Exercises

1. For the $(j=1)$ representation of the rotation group find the finite rotation matrix around the z axis, $R_z(\theta)$, in the basis where I_z is diagonal. The $(j=1)$ representation of the rotation group should be equivalent to the rotation group itself (which rotates coordinates $\{x,y,x\}$), right? But what is the matrix that rotates $\{x,y,x\}$ around the z-axis? And what is object the matrix $R_z(\theta)$ rotates? How about the general (j) representation with integer j?

2. Show that the generator $I_{\vec{n}}$ for the rotation around axis $\vec{n}$ is given as $I_{\vec{n}} = \vec{I}\cdot\vec{n}$ (note the vector dot product, not a matrix acting on a vector).
   Hint: Show that rotation around $\vec{n}$ has to leave $\vec{n}$ unchanged, and therefore $I_{\vec{n}}\vec{n} = 0$ (here is a matrix acting on a vector). Prove the last equation by direct calculation.

3. Find the finite matrix for a rotation around axis $\vec{n}$ over anble θ for a 2×2 representation of the rotation group.

   Hints:
   • $\vec{I} = \frac{1}{2}\vec{\sigma}$, and $(\vec{\sigma}\cdot\vec{n})^2 = 1$, where $\vec{\sigma}$ are Pauli matrices.

4. Argue that the $(j=\frac{1}{2})$ representation of the rotation group is $SU(2)$ – a group of 2×2 unitary matrices with determinant 1. Argue that there is a 2-to-1 homorphism $SU(2)\to SO(3)$.

5. Find the additive parameter φ of the velocity boost matrix (along the x-axis) and the corresponding generator.
   Hint: $v = \tanh\varphi$.

6. Using the additive parameter $\varphi = \frac{1}{2}\ln\frac{1+v}{1-v}$ find the velocity-boost matrices for the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of the Lorentz group.

7. Consider a classical particle with Lagrangian $\mathcal{L}(\vec{r},\vec{v})$ where $\vec{r}=\{x,y,z\}$ is the coordinate of the particle and $\vec{v}=\{\dot{x},\dot{y},\dot{z}\}$ is its velocity.
   • Suppose the Lagrangian is invariant under the transformation
   $$\vec{r}\to\vec{r}+\epsilon\vec{q}\,,\ \vec{v}\to\vec{v}+\epsilon\dot{\vec{q}}\,,$$
   where $\vec{q}$ is a certain function of time, $\dot{\vec{q}}$ is its time-derivative, and ϵ is an infinitesimally small quantity. Show that the quantity
   $$\frac{\partial\mathcal{L}}{\partial\vec{v}}\vec{q}$$
   is conserved on the trajectories of the particle.
   • Consider a particle with mass m in a spherically symmetric potential $V(r)$ with the Lagrangian
   $$\mathcal{L} = \frac{1}{2}m\vec{v}^2 - V(r)\,.$$
   Show that the rotation around the z-axis corresponds to the above transformation with
   $$\vec{q} = \{y,-x,0\}\,.$$
   Show that the corresponding conserved quantity is the z-projection of the angular momentum of the particle.

8. Suppose j^a is a conserved Noether's current. Consider a new current,
   $$j'^a = j^a + \partial_b X^{ab}\,,$$
   where X^{ab} is an arbitrary anti-symmetric tensor, $X^{ab} = -X^{ba}$. Is the new current conserved? Is the total charge, $\int d^3\vec{r} j^0$, conserved?