1. Consider the ("classical") Lagrangian density

$$\mathcal{L} = \partial_a \phi^\dagger \partial^a \phi - m^2 \phi^\dagger \phi$$

where the Lorentz scalar field ϕ is a dublet

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} .$$

Show that the Lagrangian is invariant under a certain group of (global) continuous transformations. Identify the group and find the conserved (Noether's) currents.

Hints:

(a) Consider an infinitesimal transformation

$$\varphi \to (1 + i I_k \alpha_k)\varphi .$$

Show that (for the solutions of the corresponding Euler-Lagrange equations) the variation of the Lagrangian under this transformation is given as

$$\delta\mathcal{L} = -\partial_a(\alpha_k J_k^a) ,$$

with the currents

$$J_k^a = i\left(\varphi^\dagger I_k \partial^a \varphi - \partial^a \varphi^\dagger I_k \varphi\right) .$$

Argue that for the global transformation, α_k = const, the symmetry, $\delta\mathcal{L} = 0$, of the Lagrangian leads to the conservation of these currents,

$$\partial_a J_k^a = 0 .$$

(b) Make the theory locally gauge invariant using our usual trick,

$$\partial_a \to D_a = \partial_a + i g B_a^k I_k .$$

Now find out the interaction term of the lowest order: it must have the form

$$-g J_a^k B_k^a ,$$

where J_a^k must be the conserved currents. Check that these are the same currents as in the first hint.

2. Consider the ("classical") Lagrangian density

$$\mathcal{L} = i\bar{\psi}\gamma^a \partial_a \psi - m\bar{\psi}\psi$$

where the Lorentz bispinor field ψ is an (isospin) dublet

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} .$$

Show that the Lagrangian is invariant under a certain group of continuous transformations. Identify the group and find the conserved (Noether's) currents.

3. (Extra) Argue that the non-abelian charges of a Yang-Mills theory have the same commutation relation as the generators of the symmetry group of the theory.

4. (Extra) Show that the Dirac Lagrangian is invariant under the chiral transformation,

$$\psi \to e^{i\theta\gamma_5}\psi ,$$

if the mass of the Dirac field vanishes. Find the corresponding current. For $m > 0$ evaluate the current and its divergence.