Exercises

1. Calculate the current and the momentum of the positive- and negative-frequency solutions of the Dirac equation.

2. The spinors $\phi \in (\frac{1}{2}, 0)$ and $\chi \in (0, \frac{1}{2})$ are often called "left" and "right". Explain why.

   Hint: investigate the behavior of the projection of the spin on the momentum, $\phi^\dagger \frac{1}{2}\vec{\sigma}\vec{p}\phi$, as function of the velocity with which the spinor moves relative to the observer.

   What happens if the spinor has zero-mass and is thus doomed to forever move with the speed of light?

3. Show that $\frac{d^3p}{2E_{\vec{p}}}$ is invariant.

   Hint: consider $d^4p\delta(p^2 - m^2)$.

4. Consider the parity transformation of coordinates as a similarity transformation with the diagonal transformation matrix

   $$P = \mathrm{diag}(1, -1, -1, -1)$$

   such that under parity transformation

   $$x \to Px \,.$$

   Calculate the transformed generators $P\vec{J}P^{-1}$ and $P\vec{K}P^{-1}$.

5. Argue that the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group is equivalent (up to a similarity transformation) to the the Lorentz group itself (the 4-vector representation) while the same-dimension representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ is not.

   Hint: argue that $(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ and recall the $\{\phi^\dagger\phi, \phi^\dagger\vec{\sigma}\phi\}$ exercise.

6. For our spin-$\frac{1}{2}$ field calculate the canonical equal-time anti-commutation relation,

   $$\{\psi(t, \vec{r}), \Pi(t, \vec{r}\,')\} \,, \qquad (1)$$

   where Π is the conjugate momentum,

   $$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)} \,. \qquad (2)$$

7. When we turned (by correspondingly commuting and anti-commuting) the Hamiltonians of the scalar and the Dirac fields into the canonical form (with the number-of-particle operators) we disregared some infinite contributions. What are the signs of this contributions for the scalar and for the Dirac fields?