Exercises

1. Consider a Lagrangain
$$\mathcal{L} = \bar{\psi}\gamma^a i(\partial_a + igA_a)\psi\,. \quad (1)$$
Argue that this Lagrangian is invariant uder the local gauge transformation
$$\begin{cases} \psi & \rightarrow & \psi' = U\psi\,, \\ A_a & \rightarrow & A'_a = UA_aU^{-1} - \frac{1}{ig}(\partial_a U)U^{-1}\,. \end{cases} \quad (2)$$

2. Find the infinitesimal gauge-transformation rule for the gauge field A^a_j.

3. Argue that the covariant derivative D_μ transforms under $U(1)$ gauge trasformation as
$$D_\mu \rightarrow D'_\mu = e^{i\alpha(x)}D_\mu e^{-i\alpha(x)}$$

4. Find the transformation law of the covariant derivative under a general gauge transformation $\psi \rightarrow U\psi$.

5. Find the transformation law of the Yang-Mills field tensor
$$F_{ab} = \frac{1}{ig}[D_a, D_b]$$

6. Argue that the Yang-Mills Largrangian
$$L_{YM} = -\frac{1}{2}\mathrm{Tr}(F_{ab}F^{ab})$$
is gauge invariant.

7. (Extra) Assume that for any complex numbers a and b it is valid that
$$\int_a^b z^n dz = \frac{b^{n+1} - a^{n+1}}{n+1}\,. \quad (3)$$
Now,

   (a) Argue that if the function f is analytic at the point z_0 (that is, can be represented as a Taylor-series, $f(z) = \sum_n c_n(z-z_0)^n$, in the vicinity of the point z_0) that the integral over a closed path around this point is zero,
   $$\oint_{z_0} f(z)dz = 0\,. \quad (4)$$

   (b) Argue that if the function f is analytic in some region, one can (smoothly) distort the integral path within this region without changing the value of the integral.

   (c) Argue that the inegral of $1/z$ around zero is equal $2\pi i$,
   $$\oint_0 \frac{dz}{z} = \int_0^{2\pi} \frac{rde^{i\phi}}{re^{i\phi}} = 2\pi i\,. \quad (5)$$

   (d) Argue the if the function f is analytic at the point z_0, then
   $$\oint_{z_0} dz\frac{f(z)}{z-z_0} = 2\pi i f(z_0)\,. \quad (6)$$