**Exercises: spin-$\frac{1}{2}$**

1. Show that after a rotation by $2\pi$ a spinor changes sign.
   Hint: a 'spinor' is an object which under rotations of coordinates is transformed by a ($j = \frac{1}{2}$) representation of the rotation group.

2. *Adjoint representation*
   Consider a group of objects $\{g(\alpha)\}$ with infinitesimal element
   $$g(d\alpha) = 1 + i \sum_{k=1}^{n} I_k d\alpha_k$$
   and Lie algebra
   $$I_k I_m - I_m I_k = i \sum_{l=1}^{n} C_{km}{}^{l} I_l \,.$$
   • For a given $\beta$ consider an automorphism
   $$g(\alpha) \to h(\alpha) \doteq g(\beta) g(\alpha) g(\beta)^{-1} \,,$$
   and show that the generators $J_k$ of the group $\{h\}$ are linear combinations of the generators $I_k$ of the group $\{g\}$,
   $$J_k = \sum_{l=1}^{n} A_{kl}(\beta) I_l \,,$$
   where $A_{kl}(\beta)$ are certain $n \times n$ matrices.
   • Show that matrices $A$ form a group which is a representation of the group $\{g\}$. This representation is called the adjoint representation.
   • Show that the generators $\mathcal{I}_k$ of the adjoint representation,
   $$A_{kl}(d\beta) = \delta_{kl} + i \sum_{m} (\mathcal{I}_m)_{kl} d\beta_m \,,$$
   are matrices with the matrix elements
   $$(\mathcal{I}_k)_{lm} = i C_{lk}{}^{m} = -i C_{kl}{}^{m} \,.$$
   • What is the adjoint representation of the $SU(2)$ group?

3. Let $\phi$ and $\chi$ be spinors which transforms under correspondingly $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representation of the Lorentz group. Argue that the objects $\{\phi^\dagger\phi, \phi^\dagger\vec{\sigma}\phi\}$ and $\{\chi^\dagger\chi, -\chi^\dagger\vec{\sigma}\chi\}$ transform as four-vectors under Lorentz transformation.

   Hints:
   • For simplicity consider only infinitesimal transformations,
   $$\phi \to (1 + i\frac{1}{2}\vec{\sigma} d\vec{w})\phi \,,$$
   $$\chi \to (1 + i\frac{1}{2}\vec{\sigma} d\vec{w}^*)\chi \,,$$
   and consider separately a rotation around $z$-axis,
   $$\vec{\sigma} d\vec{w} = \sigma_z d\theta \,, \quad (1)$$
   and a velocity boost in the $x$-direction,
   $$\vec{\sigma} d\vec{w} = \sigma_x i dv \,. \quad (2)$$

4. Show that the Lagrangian
   $$\mathcal{L} = \tfrac{1}{2}\left(\bar{\psi}\gamma^a \partial_a \psi + \partial_a \bar{\psi}\gamma^a \psi\right) - m\bar{\psi}\psi$$
   does not lead to a good Euler-Lagrange equation.

5. Show by direct calculation that the current $\bar{\psi}\gamma^a\psi$ conserves if $\psi$ is a solution of the Dirac equation.

6. Calculate the energy-momentum tensor for the spin-$\frac{1}{2}$ field with the Lagrangian
   $$\mathcal{L} = \bar{\psi}(i\gamma^a \partial_a - m)\psi \,. \quad (3)$$

7. Show that if bispinor $\psi$ is a solution to the Dirac equation, then its components satisfy the Klein-Gordon equation.
   Hint: multiply the Dirac equation by $(i\gamma^b\partial_b + m)$ from the left and use the anticommutator
   $$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab} \,, \quad (4)$$
   where $\eta^{ab}$ is the Minkowski metric tensor.