## Exercises: Lagrangian field theory

1. Consider a particle with the action

$$S = \int dt\mathcal{L}(\vec{r}, \vec{v}) \; ,$$

where $\vec{r}$ is the coordinate of the particle, $\vec{v}$ is the velocity of the particle, and the integral is taken along the particle's trajectory.

(a) Show that the equation of motion of the body (the Euler-Lagrange equation) is given as

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \vec{v}} = \frac{\partial \mathcal{L}}{\partial \vec{r}} \; .$$

(b) Momentum $\vec{p}$ of the particle is the quantity which is conserved along the trajectory of the particle if the Lagrangian does not depend on $\vec{r}$ (through the Noether's theorem). Argue, that

$$\vec{p} = \frac{\mathcal{L}}{\partial \vec{v}} \; .$$

(c) Energy $\mathcal{E}$ is the quantity which conserves (along the trajectory of the body) if the Lagrangian does not depend explicitely on time (through the Noether's Theorem). Indeed in this case the variation of the Lagrangian under the infinitesimal transformation $t \to t + dt$ is given as

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \vec{r}} d\vec{r} + \frac{\partial \mathcal{L}}{\partial \vec{v}} d\vec{v} \; .$$

Show that on the trajectory this can be written as the energy conservation law,

$$\frac{d\mathcal{E}}{dt} = 0 \; ,$$

with the energy

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \vec{v}} \vec{v} - \mathcal{L} \; .$$

2. Derive the equation of motion (the Newton's second law),

$$m\frac{d\mathbf{v}}{dt} = -\nabla V(\mathbf{r}) \; ,$$

for a non-relativistic particle with mass $m$ moving in a potential $V(\mathbf{r})$ from the action

$$S = \int_{t_1}^{t_2} dt \left( \frac{m\mathbf{v}^2}{2} - V(\mathbf{r}) \right) \; .$$

Find also the particle's momentum $\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \mathbf{v}}$ and energy $E = \mathbf{p} \cdot \mathbf{v} - \mathcal{L}$.

3. Consider a relativistic particle with mass $m$ and the action

$$S = -mc \int ds \, ,$$

where the integral is taken along the trajectory of the particle.

Find i) the equation of motion, ii) the momentum, ii) the energy of the particle, iii) the connection between the energy and the momentum. Write all this in 4-notation.

Show that the rest-energy (the energy in the frame where the particle is at rest) is given as

$$E_0 = mc^2 \; .$$

Hints: show that the Lagrangian is equal

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \; .$$

4. Derive the Klein-Gordon equation

$$\partial_a \partial^a \phi + m^2 \phi = 0 \; ,$$

from the Lagrangian

$$\mathcal{L} = \partial_a \phi^* \partial^a \phi - m^2 \phi^* \phi \; .$$

5. Show that for a solution $\phi$ of the Klein-Gordon equation the current $j^a = i(\phi^* \partial^a \phi - \partial^a \phi^* \phi)$ is conserved (that is, $\partial_a j^a = 0$).

6. Derive the Maxwell equations with sources,

$$\partial_a \partial^a A^b = 4\pi j^b \; ,$$

from the Lagrangian

$$\mathcal{L} = -\frac{1}{8\pi} \partial_a A^b \partial^a A_b - j^a A_a \; ,$$

with the Lorenz condition $\partial_a A^a = 0$.

7. Show that $d^4x$ and $dVj^0$ are Lorentz scalars, and that $dVT^{0a}$ is a Lorentz 4-vector.

8. Argue, from the minimal action principle, that adding a divergence to the Lagrangian, $\mathcal{L} \to \mathcal{L} + \partial_a X^a(\phi)$, where $X^a(\phi)$ are some functions of the field $\phi$, should not change the equations of motion. Check directly that the Euler-Lagrange equation indeed is the same for the two Lagrangians. Now, does the energy-momentum tensor change after the addition of the divergence? Does the total energy $\mathcal{E} = \int T_0^0 d^3x$ change? Is there a problem?