Transformational properties of fields

Group of coordinate transformations

The coordinate transformations between inertial frames form a group where the group operation is composition (performing one transformation after another). The identity element is the identical transformation and the inverse element is the inverse transformation.

In special relativity the group of admissible coordinate transformations is called the *Poincaré group*. It includes velocity boosts, rotations, translations and reflections.

We shall first consider the (restricted) *Lorentz group* – a continuous subgroup of the Poincaré group made of rotations and velocity boosts.

An n-component physical quantity ψ under a coordinate transformation from the Lorentz group,

$$x \to x' = \Lambda x \,, \tag{1}$$

should transforms via some $n \times n$ matrices $t(\Lambda)$,

$$\psi \to \psi' = t(\Lambda)\psi \,. \tag{2}$$

Since the group operations are preserved,

$$t(\Lambda_1\Lambda_2) = t(\Lambda_1)t(\Lambda_2) \,,\; t(\Lambda^{-1}) = t(\Lambda)^{-1} \,, \tag{3}$$

matrices $\{t(\Lambda)\}$ form a group *homomorphic* to the group $\{\Lambda\}$. The group $\{t(\Lambda)\}$ is referred to as a *representation* of the group of coordinate transformation.

Covariant physical quantities are transformed by a representation of the Lorentz group.

Lie groups and Lie algebras

The transformation matrices from the Lorentz group — rotations and velocity boosts — are differentiable functions of the transformation parameters, correspondingly, rotation angle, and velocity.

A group $\{g\}$ whose elements are differentiable functions of some continuous parameters, $g(\alpha_1, ..., \alpha_n)$, is called a *Lie group*.

An element close to unity, assuming $g(0) = \mathbf{1}$, can be expressed in terms of the *generators* I_k,

$$g(d\alpha) = 1 + i\sum_{k=1}^{n} I_k d\alpha_k \,. \tag{4}$$

The commutation relation for the generators is called the *Lie algebra*,

$$I_j I_m - I_m I_j = i\sum_{k=1}^{n} C^k_{jm} I_k \,, \tag{5}$$

where C^k_{jm} are the *structure constants*.[1]

A representation of a group has the same Lie algebra as the group itself. The Lie algebra largely defines the properties of a Lie group and its representations.

Lie algebra of rotation group

The transformation of the coordinates under a rotation around z-axis over the angle θ is given as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,. \tag{9}$$

For an infinitesimally small angle $d\theta$ the rotation matrix can be written as

$$1 + i\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\theta \equiv 1 + iI_z d\theta \,, \tag{10}$$

where I_z is the generator of an infinitesimal rotation around z-axis. The corresponding generators I_x and I_y are

$$I_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} ,\; I_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \,. \tag{11}$$

Direct calculation shows that these generators have the Lie algebra

$$I_k I_l - I_l I_k = i\sum_m \epsilon_{klm} I_m \,, \tag{12}$$

where ϵ_{klm} is the antisymmetric symbol[2] (also called Levi-Civita symbol) of rank 3.

Rotation group is customarily denoted $SO(3)$[3].

Lie algebra of the Lorentz group

The Lorentz group consists of rotations and velocity boosts. The 4-dimensional rotation generators can

[1] Consider the equation

$$g(\beta') = g(\alpha)g(\beta)g^{-1}(\alpha) \,. \tag{6}$$

It defines a function $\beta'(\beta, \alpha)$ with the property $\beta'(0, \alpha) = 0$. Making infinitesimal expansion of the equation and collecting the terms of the order $\alpha\beta$ gives

$$\sum_{klm} iI_k \frac{\partial^2 \beta'_k}{\partial\beta_l \partial\alpha_m} \beta_l \alpha_m = \sum_{lm} (I_l I_m - I_m I_l)\beta_l \alpha_m \,, \tag{7}$$

from which follows (5) where

$$C^k_{lm} = \frac{\partial^2 \beta'_k}{\partial\beta_l \partial\alpha_m} \,. \tag{8}$$

[2] ϵ_{jkl} is equal +1 (-1) if (j, k, l) is an even (odd) permutation of $(1, 2, 3)$, otherwise it is equal zero.

[3] Special (det = 1), Orthogonal, 3 × 3.

be written immediately from (10) and (11) as

$$J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} . \tag{13}$$

The Lorentz boost matrix for an infinitesimally small relative velocity dv along one of the axes is

$$\begin{pmatrix} 1 & -dv \\ -dv & 1 \end{pmatrix} = 1 + i \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} dv . \tag{14}$$

Thus the three generators of velocity boosts are

$$K_z = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad K_y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_x = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{15}$$

Direct calculation gives the Lie algebra of the Lorentz group,

$$J_k J_l - J_l J_k = i \sum_m \epsilon_{klm} J_m \tag{16}$$

$$J_k K_l - K_l J_k = i \sum_m \epsilon_{klm} K_m \tag{17}$$

$$K_k K_l - K_l K_k = -i \sum_m \epsilon_{klm} J_m . \tag{18}$$

The infinitesimal group element corresponding to a rotation around direction $\vec{n}$ with angle $d\theta$, and a velocity boost $d\vec{v}$, is given as[4]

$$g = \mathbf{1} + i\vec{J}\vec{n}d\theta + i\vec{K}d\vec{v} . \tag{19}$$

The Lie algebra of the Lorentz group can be written in a more symmetric way with the (complex) parameterization

$$d\vec{w} = \vec{n}d\theta + id\vec{v} . \tag{20}$$

The infinitesimal Lorentz transformation is then

$$g = 1 + i\vec{M}d\vec{w} + i\vec{N}d\vec{w}^* , \tag{21}$$

where the (hermitian) generators $\vec{M}$ and $\vec{N}$ are linear combinations of generators $\vec{J}$ and $\vec{K}$

$$\vec{M} = \frac{1}{2}(\vec{J} - i\vec{K}) \; , \; \vec{N} = \frac{1}{2}(\vec{J} + i\vec{K}) . \tag{22}$$

[4] where $\vec{a}\vec{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3$

The Lie algebra for the new generators is

$$M_k M_l - M_l M_k = i \sum_m \epsilon_{klm} M_m , \tag{23}$$

$$N_k N_l - N_l N_k = i \sum_m \epsilon_{klm} N_m , \tag{24}$$

$$M_k N_l - N_l M_k = 0 . \tag{25}$$

Thus the Lorentz Lie algebra is a combination of two independent rotation Lie algebras.

Irreducible representations of the rotation group

Instead of the generators I_x, I_y, I_z we shall use another parameterization, I_+, I_-, I_z, where

$$I_\pm = \frac{1}{\sqrt{2}}(I_x \pm iI_y), \tag{26}$$

with the commutation relations

$$I_z I_\pm - I_\pm I_z = \pm I_\pm \; , \; I_+ I_- - I_- I_+ = I_z . \tag{27}$$

Since I_z is hermitian, it has a set of real eigenvalues λ with the corresponding eigenvectors $|\lambda\rangle$,

$$I_z|\lambda\rangle = \lambda|\lambda\rangle . \tag{28}$$

From the commutation relations (27) it follows that the states $I_\pm|\lambda\rangle$ are also eigenvectors of I_z,

$$I_z(I_\pm|\lambda\rangle) = (\lambda \pm 1)(I_\pm|\lambda\rangle). \tag{29}$$

For a finite dimension representation there must exist the largest eigenvalue, say j, such that

$$I_+|j\rangle = 0. \tag{30}$$

Similarly, there should also exist the smallest eigenvalue, such that

$$(I_-)^{(N+1)}|j\rangle = 0, \tag{31}$$

where N is some integer number.

Thus the eigenvalues of I_z is the sequence

$$j, j-1, j-2, ..., j-N. \tag{32}$$

The trace of the generator I_z is equal zero[5],

$$\begin{aligned} \mathrm{trace}(I_z) &= j + (j-1) + ... + (j-N) \\ &= \frac{1}{2}(2j-N)(N+1) = 0 . \end{aligned} \tag{33}$$

Thus $j = N/2$ (either integer or half-integer) and the eigenvalues of I_z are $j, j-1, ..., -j$. The dimension of a representation with a given j is $2j+1$.

The quantities that transform under the irreducible representations (j) of the rotation group are the spherical tensors which are irreducible combinations of normal Euclidean tensors. Half integer j correspond to objects called *spinors*.

[5] taking trace of the commutation relation