Exercises

1. Consider two interacting fields, a complex spin-$\frac{1}{2}$ field ψ and a real scalar field ϕ with interacting Lagrangian $\mathcal{L}_v = -g\bar{\psi}\psi\phi$. Find the Euler-Lagrange equations and the Hamiltonian density T_0^0 of the system.

2. For a real scalar field $\mathcal{L} = \frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{2}m^2\phi^2$ show that commutation relations $[a_k, a_k^\dagger] = 1$ for the generation/annihilation operators lead to the commutation relation $[\chi, \pi] = i$ for the canonical generalized coordinate χ and the corresponding generalized momentum π. Hints:

   (a) The generalized coordinates of a field are the values of the field itself, $\chi = \phi$. According to the general rule the generalized momentum is $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\chi}}$. Find the expression for π. (Answer: $\pi = \partial_0\phi$).

   (b) Consider the amplitude of a single plane wave as the generalized coordinate (now a quantum-mechanical operator)

   $$\chi = \phi_k = \frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}e^{-ikx} + a_{\vec{k}}^\dagger e^{+ikx}\right)$$

   and calculate the corresponding operator π.

   (c) Calculate the commutator $[\chi, \pi]$.

3. Time evolution in Heisenberg-Born-Jordan matrix mechanics: show that the canonical Hamilton equations of motion[1]

   $$\frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}\,,\ \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p}\,,\ \frac{\partial f(q,p)}{\partial t} = \{f, H\}\,,$$

   correspond to operator equations

   $$\frac{\partial p}{\partial t} = \frac{1}{i}[p, H]\,,\ \frac{\partial q}{\partial t} = \frac{1}{i}[q, H]\,,\ \frac{\partial f}{\partial t} = \frac{1}{i}[f, H]\,.$$

   with the commutation relation $[q, p] = i$

   Hint: show (by induction) that

   $$\frac{1}{i}[q, f] = \frac{\partial f}{\partial p}\,,\ \frac{1}{i}[p, f] = -\frac{\partial f}{\partial q}$$

   and

   $$\frac{1}{i}[h, f] = \frac{\partial h}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial f}{\partial q}\frac{\partial h}{\partial p}$$

   for all functions f and g that can be represented as a series of powers of q and p.

[1] where

$$\{f, H\} \equiv \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}$$

is the classical Poison bracket.