Exercises: special relativity

1. Argue that coordinate transformations between inertial frames of reference form a mathematical group.

2. Derive the Lorentz transformation, particularly, the velocity boost in x-direction,

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{bmatrix} 1 & \frac{-v}{c^2} \\ -v & 1 \end{bmatrix} \begin{pmatrix} t \\ x \end{pmatrix},$$

   using group postulates and the *locality of interactions* (in other words, the existence of a maximal speed of information transmission).

3. Show that in the non-relativistic limit, $v \ll c$, Lorentz transformations reduce to Galilean transformations.

4. Consider a composition of two Galilean transformations and derive the classical velocity-addition formula.

5. Consider a composition of two Lorentz transformations and derive the relativistic velocity-addition formula.

6. Formulate the relativistic velocity-addition formula for the *rapidity* ϕ defined as

$$\tanh(\phi) = \frac{v}{c} .$$

   Hint: use the law of addition of hyperbolic tangents,

$$\tanh(a+b) = \frac{\tanh(a)+\tanh(b)}{1+\tanh(a)\tanh(b)} .$$

7. Show that the *infinitesimal interval*

$$ds^2 = c^2dt^2 - d\mathbf{r}^2 .$$

   is invariant under a general Lorentz transformation (that is, boosts and spatial rotations). Show that the interval can also be written as

$$ds^2 = dx_a dx^a \equiv dx^a dx^b g_{ab} .$$

   Show that in Minkowski space the finite interval,

$$\Delta s^2 = c^2\Delta t^2 - \Delta\mathbf{r}^2 .$$

   is also invariant.

8. Show that the group of Lorentz transformations is $SO(1,3)$.

9. Show that a moving clock runs slower, than stationary. Hint: consider the transformation of

$$\begin{pmatrix} dt \\ dx = 0 \end{pmatrix}.$$

10. Show that a moving rod is shorter, than stationary. Hint: consider a transformation into

$$\begin{pmatrix} dt' = 0 \\ dx' \end{pmatrix}.$$

11. Which of the following objects are covariant?
    - Kronecker delta $\delta^a_b \equiv \{a == b?1 : 0\}$;
    - Lorentz transformation matrix Λ^a_b;
    - Metric tensor g_{ab}

12. LHC shall collide two 7 TeV proton beams which travel in the opposite directions. What is the veclocity of protons with respect to Earth? What is the relative velocity of the colliding protons?

13. The Greisen–Zatsepin–Kuzmin limit (GZK limit) is a theoretical upper limit on the energy of cosmic rays (high energy charged particles from space) coming from "distant" sources. The limit is set by interactions of cosmic ray protons with the microwave background radiation over long distances (∼160 million light-years).

    (a) The universe is a blackbody at 2.73 K. What is the average energy of a cosmic microwave background photon (in eV) in the outer space?

    (b) How much energy would a proton (p^+) need to collide with a cosmic microwave background photon (γ_{CMB}) in outer space to generate a 135 MeV pion (π^0)? In other words, what is the energy threshold for $p^+ + \gamma_{CMB} \to p^+ + \pi^0$?

    (c) Why should we not see cosmic rays (high energy protons) from distant sources with energy above certain limit? What is the limit?