

Exercises: free scalar field

1. For a scalar field prove the orthogonality of plane waves with periodic boundary condition.
2. For the Lagrangian

$$\mathcal{L} = \partial_a \phi^* \partial^a \phi - m^2 \phi^* \phi$$

find the expressions for the energy-momentum tensor, T_b^a , and the conserved current density, j^a .

3. Calculate the 4-momentum,

$$P^a = \int_V dV T^{0a},$$

and the conserved current,

$$J^a = \int_V dV j^a,$$

for positive and negative frequency normal modes. Interpret the results.

4. Prove that different normal modes contribute to the total energy and the total charge of a scalar field independently (that is, there are no interference terms in the sums).
5. Given $aa^\dagger - a^\dagger a = 1$ ($aa^\dagger + a^\dagger a = 1$) find the eigenvalues of the $a^\dagger a$ operator.

Hint: consider the eigenstates of the $a^\dagger a$ operator,

$$a^\dagger a |\lambda\rangle = \lambda |\lambda\rangle,$$

and investigate the states $a|\lambda\rangle$ and $a^\dagger|\lambda\rangle$.