Exercises

1. Prove that generators of a Lie group have a Lie algebra

$$[I_l, I_m] \doteq I_l I_m - I_m I_l = iC^k_{lm} I_k \ ,$$

where C^k_{lm} are certain numbers (called structure constants).

Hints:

Consider the equation

$$g(\beta') = g(\alpha)g(\beta)g^{-1}(\alpha)$$

in the limit $\beta \to 0$ and $\alpha \to 0$ and argue that

$$\lim_{\beta\to 0} \beta'_k = f_{kl}(\alpha)\beta_l \ ,$$

where $f_{kl}(\alpha)$ are some functions.

Argue that

$$\lim_{\alpha\to 0} f_{kl}(\alpha) = \delta_{kl} + \sum_m C^k_{lm}\alpha_m \ ,$$

where C^k_{lm} are some constants.

Now consider the infinitesimal form of the equation.

2. Argue that a representation of a Lie group is also a Lie group with the same Lie algebra, as the original group.

3. Show that if the operators

$$U = \exp\left(-i\sum_k J_k\alpha_k\right)$$

are unitary, $U^\dagger U = 1$, and the parameters α_k are real, then the generators J_k are hermitian: $J_k^\dagger = J_k$. Show that if $\det(U) = 1$ then $\mathrm{trace}(J_k) = 0$.

Hints:
– consider only infinitesimal operators;
– show that $\det(e^A) = e^{\mathrm{trace}(A)}$ by diagonalizing matrix A.

4. Using the Lie algebra of the rotation group[1]

$$[I_j, I_k] = i\sum_l \epsilon_{jkl} I_l$$

find the 2x2 representation of the rotation generators $\vec{I}$ (assuming I_3 is diagonal and I_1 is real).

Hints: recall that $\mathsf{trace}(I_k) = 0$ and $I_k^+ = I_k$; I's must be half the Pauli's σ-matrices.

[1] ϵ_{jkl} is the Levi-Civita (absolutely antisymmetric) symbol.