Exercises: spin-$\frac{1}{2}$

1. Show that after a rotation by 2π a spinor changes sign.
   Hint: a ‘spinor’ is an object which under rotations of coordinates is transformed by a $j = \frac{1}{2}$ representation of the rotation group.

2. An *adjoint representation* of a group $\{g(\alpha)\}$ with infinitesimal element

$$g(d\alpha) = 1 + i \sum_{k=i}^{n} I_k d\alpha_k$$

   and Lie algebra

$$I_k I_m - I_m I_k = i \sum_{l} C_{km}{}^{l} I_l$$

   is the matrix representation $\{G(\alpha)\}$ (where group elements G are $n \times n$ matrices) where one considers transformations of the group elements in the form[1]

$$g \to g' = g(\alpha)^{-1} g g(\alpha) \,.$$

   Specifically, matrices G transform the group generators,

$$I_k \to g(\alpha)^{-1} I_k g(\alpha) = \sum_{l} G_{kl}(\alpha) I_l \,.$$

   Consider an infinitesimal transformation and show that the generators $\mathcal{I}_k$ of the adjoint representation,

$$G_{kl}(\alpha) = \delta_{kl} + i \sum_{m} (\mathcal{I}_m)_{kl} \alpha_m$$

   are matrices with the matrix elements

$$(\mathcal{I}_k)_{lm} = i C_{lk}{}^{m} = -i C_{kl}{}^{m} \,.$$

3. Let ϕ be a spinor which transforms under $(\frac{1}{2}, 0)$ representation of the Lorentz group. Argue that the object $\{\phi^\dagger\phi, \phi^\dagger\vec{\sigma}\phi\}$ transforms as four-vector under Lorentz transformation.

   Hints:

   - For simplicity consider only infinitesimal transformations;
   - First consider the transformation of the object under rotations: argue (form the previous exercise or from equations (10)-(11) in note 4), that for ordinary three-vectors
   $$(I_k)_{lm} = -i\epsilon_{klm}.$$
   - Consider velocity boost in the $\vec{n}$ (or, for simplicity, in the x) -direction.

4. Show that the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\bar{\psi}\gamma^a\partial_a\psi + \partial_a\bar{\psi}\gamma^a\psi\right) - m\bar{\psi}\psi$$

   does not lead to a good Euler-Lagrange equation.

5. Show by direct calculation that the current $\bar{\psi}\gamma^a\psi$ conserves if ψ is a solution of the Dirac equation.

6. Calculate the energy-momentum tensor for the spin-$\frac{1}{2}$ field.

7. Show that if bispinor ψ is a solution to the Dirac equation, then its components satisfy the Klein-Gordon equation.
   Hint: multiply the Dirac equation by $(i\gamma^b\partial_b + m)$ from the left.

8. Argue that the 2×2 representation of the Lie algebra of the rotation group is the group of Special Unitary 2×2 matrices (called $SU(2)$). In other words, argue that $SU(2)$ group has the same Lie algebra, as the rotation group $SO(3)$.

[1] adjoint representation is actually dual: one can consider both $g(\alpha)^{-1}gg(\alpha)$ and $g(\alpha)gg(\alpha)^{-1}$.