Exercises: Lagrangian field theory

1. Consider a particle with the action

$$S = \int dt L(q, \dot{q}) ,$$

where q is the (generalized) coordinate, $\dot{q}$ is the (generalized) velocity, and the integral is taken along the particle's trajectory.

   (a) Derive the eqation of motion for the particle.

   (b) Consider infinitesimal translations and derive the energy-momentum conservation law.

2. Derive the Newton's second law of motion for a particle with mass m moving in a potential $V(\mathbf{r})$,

$$m\frac{d\mathbf{v}}{dt} = -\nabla V(\mathbf{r}) ,$$

from the action

$$S = \int_{t_1}^{t_2} dt \left(\frac{m\mathbf{v}^2}{2} - V(\mathbf{r}) \right) .$$

Find also the particle's momentum $\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$ and energy $E = \mathbf{p} \cdot \mathbf{v} - L$.

3. Derive the famous expression,

$$E = mc^2 ,$$

for the rest-energy of a particle with mass m from the action

$$S = -mc \int ds$$

where the integral is taken along the trajectory of the particle.

Show that $E^2 = m^2c^4 + p^2c^2$.

Hints: show that the Lagrangian is equal

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} .$$

4. Derive the Klein-Gordon equation

$$\partial_a \partial^a \phi + m^2 \phi = 0 ,$$

from the Lagrangian

$$\mathcal{L} = \partial_a \phi^* \partial^a \phi - m^2 \phi^* \phi .$$

5. Show that for a solution ϕ of the Klein-Gordon equation the current $j^a = i(\phi^* \partial^a \phi - \partial^a \phi^* \phi)$ is conserved (that is, $\partial_a j^a = 0$).

6. Derive the Maxwell equations with sources,

$$\partial_a \partial^a A^b = 4\pi j^b ,$$

from the Lagrangian

$$L = -\frac{1}{8\pi} \partial_a A^b \partial^a A_b - j^a A_a ,$$

with the Lorenz condition $\partial_a A^a = 0$.

7. Show that d^4x and dVj^0 are Lorentz scalars, and that dVT^{0a} is a Lorentz 4-vector.

8. Argue, from the minimal action principle, that adding a divergence to the Lagrangian, $\mathcal{L} \to \mathcal{L} + \partial_a X^a(\phi)$, where $X^a(\phi)$ are some functions of the field ϕ, should not change the equations of motion. Check directly that the Euler-Lagrange equation indeed is the same for the two Lagrangians. Now, does the energy-momentum tensor change after the addition of the divergence? Does the total energy $\mathcal{E} = \int T^0_0 d^3x$ change?