Classical field theory

Principle of least action

The law of motion for a classical physical system is often written as a differential equation. The solution to this equation (with appropriate boundary conditions) provides the trajectory which the system follows.

Equivalently, the laws of motion can be formulated using a functional of the trajectories—called *action* and denoted as S—through the *principle of least action* (also called *variational principle*). In this formulation the system follows the trajectory which the least action. Specifically, the principle of least action states that

> *Of all possible trajectories a physical system follows the one which provides the least (or, more strictly, stationary) action.*

or, alternatively,

> *The variation of the action vanishes on the trajectory actually followed by the physical system.*

Thus, if $\phi(x^a)$ is the actual trajectory, and $\delta\phi(x^a)$ is a small but otherwise arbitrary variation of the trajectory (which does not change the boundary condition), the (linear part of the) variation of the action vanishes,

$$\delta S(\phi) \equiv S(\phi+\delta\phi) - S(\phi) = 0. \tag{1}$$

Variational formulation has certain practical advantages over the differential formulation.

The action $S(\phi)$ is a functional—it takes a trajectory as an argument and returns a real number as a result—of the field ϕ and its derivatives $\partial_a\phi$, usually written in the form of an integral,

$$S(\phi) = \int_{\text{some volume}} d^4x \mathcal{L}(\phi, \partial_a\phi), \tag{2}$$

where the kernel $\mathcal{L}(\phi, \partial_a\phi)$ is called the Lagrangian density or simply the *Lagrangian*.

The differential equations of motion can be readily obtained as the Euler-Lagrange equations of the corresponding action.

Euler-Lagrange equation

The trajectory with minimal action satisfies a differential equation of motion,

$$\partial_a \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)} = \frac{\partial\mathcal{L}}{\partial\phi}\ , \tag{3}$$

called the Euler-Lagrange equation of the Lagrangian $\mathcal{L}$.

Indeed, the variation of the action is given as

$$\begin{aligned}\delta S(\phi) &= \int_\Omega d^4x\ \delta\mathcal{L}(\phi, \partial_a\phi)\\ &= \int_\Omega d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\delta\partial_a\phi\right)\ ,\end{aligned} \tag{4}$$

where Ω is a volume with a given boundary condition on its surface. The second term in the brackets can be integrated by parts using

$$\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\delta\partial_a\phi = \partial_a\left(\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\delta\phi\right) - \partial_a\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\delta\phi\ . \tag{5}$$

The integral of the term with full derivative can be transformed into an integral over a (hyper-)surface of the integration volume Ω. However, on the surface of the volume the variation of the trajectory vanishes as all trial trajectories satisfy the same boundary condition. Therefore the full derivative term does not contribute to the variation of the action.

The remaining terms then give

$$\delta S = \int_\Omega d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_a\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\right)\delta\phi\ . \tag{6}$$

Since the variation $\delta\phi$ is arbitrary, the variation of the action can vanish only if the expression inside the parenthesis is identically equal zero, which gives the sought equation (3).

Translation invariance and energy-momentum tensor

Experiments show that the space-time of special relativity is uniform or, in other words, *translation invariant*. Therefore the Lagrangian does not depend explicitly on coordinates x^a. This invariance (or *symmetry*) of the Lagrangian leads to a conservation law – the conservation of energy and momentum.

Under an infinitesimal translation $x^a \to x^a + \delta x^a$ the Lagrangian varies only via the fields,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_b\phi)}\partial_b\delta\phi\ . \tag{7}$$

Substituting $\partial\mathcal{L}/\partial\phi$ from the equations of motion (3) gives

$$\partial_b\left(\frac{\partial\mathcal{L}}{\partial(\partial_b\phi)}\delta\phi\right) - \delta\mathcal{L} = 0\ , \tag{8}$$

which, divided by δx^a, leads to

$$\partial_b\left(\frac{\partial\mathcal{L}}{\partial(\partial_b\phi)}\partial_a\phi\right) - \partial_a\mathcal{L} = 0\ . \tag{9}$$

which in turn can be rewritten in a form of a 4-divergence

$$\partial_b T_a^b = 0 \ , \tag{10}$$

where T_a^b is the *canonical energy-momentum tensor*,

$$T_a^b = \frac{\partial \mathcal{L}}{\partial(\partial_b \phi)} \partial_a \phi - \mathcal{L} \delta_a^b \ . \tag{11}$$

Conservation of energy and momentum

The vanishing 4-divergence can be rewritten in a 3-notation as

$$\frac{\partial}{\partial t} T_a^0 = -\sum_{i=1}^{3} \frac{\partial}{\partial r^i} T_a^i \ , \tag{12}$$

Suppose we have an isolated system contained in a (large) 3-volume V where the field is zero at the surface of the volume (and beyond). Integrating (12) over the 3-volume V gives four conservation laws for the energy and momentum[1],

$$\frac{\partial}{\partial t} P_a = 0 \ , \tag{13}$$

where P_a can be interpreted as the energy-momentum 4-vector of the system,

$$P_a = \int d^3x T_a^0 \ . \tag{14}$$

The time-component of the energy-momentum vector is the energy of the system which, when expressed via the fields, is called the *Hamiltonian* of the system,

$$H = \int d^3x T_0^0 \ . \tag{15}$$

Global gauge invariance and conserved current

If the field ϕ is complex there is another transformation which leaves the Lagrangian invariant – the global change of the phase of the field,

$$\phi \to e^{i\theta}\phi \approx \phi + i\theta\phi, \tag{16}$$

where θ is an arbitrary phase shift (infinitesimally small for simplicity). This transformation is called the *global gauge transformation*.

The variation of the Lagrangian under the phase shift is zero, since it is necessarily real[2],

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_a\phi)}\partial_a\delta\phi \\ &+ \frac{\partial \mathcal{L}}{\partial \phi^*}\delta\phi^* + \frac{\partial \mathcal{L}}{\partial(\partial_a\phi^*)}\partial_a\delta\phi^* = 0 \ . \end{aligned} \tag{17}$$

[1] note that energy and momentum are only conserved on the solutions of the Euler-Lagrange equation.

[2] and assumed to contain only integer powers of fields.

Using equations of motion (3) and the fact that $\delta\phi = i\theta\phi$, $\delta\phi^* = -i\theta\phi^*$ equation (17) can be rewritten as a continuity equation,

$$\partial_a j^a = 0 \ , \tag{18}$$

where the vector

$$j^a = \frac{1}{i}\left(\frac{\partial \mathcal{L}}{\partial(\partial_a\phi)}\phi - \frac{\partial \mathcal{L}}{\partial(\partial_a\phi^*)}\phi^* \right) \ . \tag{19}$$

is called the *conserved current*.

Conserved currents feature prominently in gauge theories as gauge fields can only couple to conserved currents.

Charge conservation

The continuity equation (18) can be interpreted as the charge conservation law. In 3-notation, $j^a = \{\rho, \mathbf{j}\}$, the continuity equation reads

$$\frac{d\rho}{dt} = -\mathrm{div}\mathbf{j} \ , \tag{20}$$

and in the integral form

$$\frac{dQ}{dt} = -\oint_{\partial V} \mathbf{j} \cdot d\mathbf{s} \ , \tag{21}$$

where Q is the total charge in the volume V,

$$Q = \int_V dV j^0 \ . \tag{22}$$

Equation (21) states that the change of the total charge in a volume can only happen via the currents through the volume's surface. The total charge of an isolated system is then conserved.

Note that for a real (that is, not complex) field the current is apparently zero: real fields, like the electromagnetic field, are necessarily neutral.

Noether's theorem

The two conservation laws, (10) and (18), are in fact manifestations of the general *Noether's theorem* which states that any differentiable symmetry of the action results in a certain conservation law. The corresponding conserved current is called *Noether's current*.