Non-relativistic limit of QFT

In the non-relativistic limit the QFT expression for the S-matrix reduces to the Lippmann-Schwinger equation for the scattering amplitude. The Lippmann-Schwinger equation is equivalent to the Schrödinger equation.

In the first order Born approximation the elastic scattering amplitude is given by the Fourier transform of the potential. Inversely, the potential is given as inverse Fourier transform of the scattering amplitude. The prescription to obtain the non-relativistic potential from a QFT is then relatively simple: calculate the relativistic transition amplitude for the elastic scattering in the lowest order perturbation theory; make the non-relativistic limit; and calculate the inverse Fourier transform.

Lippmann-Schwinger equation

Let us consider a non-relativistic elastic scattering in a system of two particles described by the Hamiltonian

$$H = H_0 + V \ , \ H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{r}^2} \ , \tag{1}$$

where $V(\mathbf{r})$ is the interaction potential, m is the reduced mass of the two particles, $\mathbf{r}$ is their relative distance, and $\hbar$ is the Planck's constant.

In the non-relativistic quantum mechanics calculation of the cross-section involves solving the the Schrödinger equation

$$(E_k - H_0)|\psi^{(+)}_{\mathbf{k}}\rangle = V|\psi^{(+)}_{\mathbf{k}}\rangle \tag{2}$$

where $E_{\mathbf{k}} = \frac{\hbar^2\mathbf{k}^2}{2m}$ and $\psi^{(+)}_{\mathbf{k}}$ is the scattering wave-function with the following asymptotics in coordinate space,

$$\psi^{(+)}_{\mathbf{k}}(\mathbf{r}) \stackrel{r\to\infty}{\longrightarrow} e^{i\mathbf{k}\mathbf{r}} + f\frac{e^{+ikr}}{r} \ . \tag{3}$$

where f is the scattering amplitude which determines the differential cross section $\frac{d\sigma}{d\Omega}$ through

$$\frac{d\sigma}{d\Omega} = |f|^2 \ . \tag{4}$$

Multiplying the Schrödinger equation from the left by the Green's function

$$G^0_E \doteq \frac{1}{E - H_0} \ , \tag{5}$$

leads to the so-called Lippmann-Schwinger equation for the scattering wave-function,

$$|\psi^{(+)}_{\mathbf{k}}\rangle = |\mathbf{k}\rangle + G^{0(+)}_{E_{\mathbf{k}}} V|\psi^{(+)}_{\mathbf{k}}\rangle \ , \tag{6}$$

where the superscript $(+)$ at the Green's function means that it has only out-going waves and $|\mathbf{k}\rangle$ is the free plane-wave solution with momentum $\mathbf{k}$,

$$H_0|\mathbf{k}\rangle = \frac{\hbar^2\mathbf{k}^2}{2m}|\mathbf{k}\rangle \ . \tag{7}$$

The Green's function in coordinate space is given as[1]

$$\begin{aligned}
\langle \mathbf{r}'|G^{0(+)}_{E_{\mathbf{k}}}|\mathbf{r}\rangle &= \sum_{\mathbf{q}\mathbf{q}'}\langle\mathbf{r}'|\mathbf{q}\rangle\frac{\delta_{\mathbf{q}\mathbf{q}'}}{\frac{\hbar^2k^2}{2m} - \frac{\hbar^2q^2}{2m}}\langle\mathbf{q}'|\mathbf{r}\rangle \\
&= -\frac{2m}{\hbar^2}\int\frac{d^3q}{(2\pi)^3}\frac{e^{i\mathbf{q}(\mathbf{r}-\mathbf{r}')}}{q^2-k^2} \\
&= -\frac{2m}{\hbar^2}\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \\
&\stackrel{r\gg r'}{\longrightarrow} -\frac{2m}{\hbar^2}\frac{1}{4\pi}\frac{e^{ikr}}{r}e^{-i\mathbf{k}'\mathbf{r}'} \ ,
\end{aligned} \tag{9}$$

where $\mathbf{k}' = \frac{k\mathbf{r}}{r}$.

The Lippmann-Schwinger equation in coordinate space is then given as

$$\begin{aligned}
\psi^{(+)}_{\mathbf{k}}(\mathbf{r}) &= e^{i\mathbf{k}\mathbf{r}} \\
&- \frac{m}{2\pi\hbar^2}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}V(r')\psi^{(+)}_{\mathbf{k}}(\mathbf{r}') \ .
\end{aligned} \tag{10}$$

Asymptotically,

$$\begin{aligned}
\psi^{(+)}_{\mathbf{k}}(\mathbf{r}) &= e^{i\mathbf{k}\mathbf{r}} \\
&- \frac{e^{ikr}}{r}\frac{m}{2\pi\hbar^2}\int d^3r' e^{-i\mathbf{k}'\mathbf{r}'}V(r')\psi^{(+)}_{\mathbf{k}}(\mathbf{r}') \ .
\end{aligned} \tag{11}$$

The factor in front of the spherical wave is the scattering amplitude,

$$\begin{aligned}
f &= -\frac{m}{2\pi\hbar^2}\int d^3r' e^{-i\mathbf{k}'\mathbf{r}'}V(r')\psi^{(+)}_{\mathbf{k}}(\mathbf{r}') \\
&= -\frac{m}{2\pi\hbar^2}\langle\mathbf{k}'|V|\psi^{(+)}_{\mathbf{k}}\rangle \ .
\end{aligned} \tag{12}$$

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[1]

$$\begin{aligned}
&\int\frac{d^3q}{(2\pi)^3}\frac{e^{i\mathbf{q}\mathbf{r}}}{q^2-p^2} \\
=&\frac{1}{(2\pi)^3}\int_0^\infty\frac{q^2dq}{q^2-p^2}\int_0^{2\pi}d\phi\int_{-1}^1 d(\cos\theta)e^{iqr\cos\theta} \\
=&\frac{1}{(2\pi)^3}\int_0^\infty\frac{q^2dq}{q^2-p^2}2\pi\left(\frac{e^{iqr}}{iqr} - \frac{-e^{iqr}}{iqr}\right) \\
=&\frac{1}{4\pi^2}\int_{-\infty}^\infty\frac{q^2dq}{q^2-p^2}\frac{e^{iqr}}{iqr} \\
=&\frac{-i}{4\pi^2}\frac{1}{r}\int_{-\infty}^\infty\frac{qdqe^{iqr}}{[q-(p+i0)][q+(p+i0)]} \\
=&\frac{-i}{4\pi^2}\frac{1}{r}2\pi i\left(\frac{qe^{iqr}}{q+p}\right)_{q=p} = \frac{1}{4\pi}\frac{e^{ipr}}{r}
\end{aligned} \tag{8}$$

Now let us introduce the scattering matrix M, defined as

$$\langle \mathbf{k}' | M | \mathbf{k} \rangle \doteq \langle \mathbf{k}' | V | \psi_{\mathbf{k}}^{(+)} \rangle = -\frac{2\pi\hbar^2}{m} f \; . \qquad (13)$$

Multiplying (6) by $\langle \mathbf{k}' | V$ from the left leads to the Lippmann-Schwinger equation for the reaction matrix M,

$$M = V + VG^0 M \; . \qquad (14)$$

The solution of the Lippmann-Schwinger equation can be written as perturbation series in V,

$$M = V + VG^0 V + VG^0 V G^0 V + \dots \; , \qquad (15)$$

where the first term is the first Born approximation,

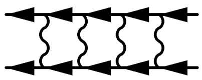
$$\langle \mathbf{k}' | M^{(1)} | \mathbf{k} \rangle = \langle \mathbf{k}' | V | \mathbf{k} \rangle = \int d^3 r V(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}} \; , \qquad (16)$$

where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ is the transferred momentum.

It is actually the series (15) that the QFT expression for the reaction amplitude reduces to in the non-relativistic limit.

Ladder diagrams

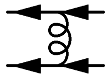
One can show that only the so called ladder diagrams,



, (17)

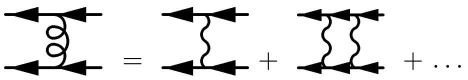
survive in the non-relativistic limit.

The total S-matrix, denoted as



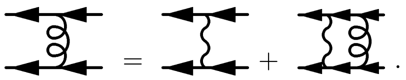
(18)

is apparently the sum of all ladder diagrams,



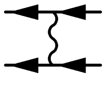
(19)

Equation (19) can be rewritten into the Lippmann-Schwinger equation,



(20)

Clearly (20) is equivalent to the Lippmann-Schwinger equation (14) if we assume that the Fourie transform of the potential is proportional to the (non-relativistic limit of the) lowest order diagram,



. (21)

Let us introduce the scattering matrix $\mathcal{M}$ via

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta^{(4)}(P_f - P_i)\mathcal{M}_{fi} \; , \qquad (22)$$

where P_i and P_f are the total initial and final momenta, and the subscrips fi denote the matrix element between the final and the initial states. With this notation the potential is given as

$$\langle \mathbf{k}' | V | \mathbf{k} \rangle = -\mathcal{M}_{fi} \; , \qquad (23)$$

where $\mathcal{M}$ is calculated from the diagram (21) in the non-relativistic limit.

In conclusion, in the non-relativistic limit a QFT reduces to the Schrödinger equation with the potential determined by the inverse Fourier transform of the lowest order elastic scattering amplitude.