Introduction

Quantum Field Theory—in the context of particle physics—is a theory of elementary particles and their interactions. The *Standard Model* of elementary particles is a quantum field theory.

By definition, elementary particles are the most fundamental—structureless—particles (like electrons and photons). They exhibit wave-particle duality: on the one hand they diffract and interfere as waves (fields), on the other hand they appear and disappear as whole entities, called quanta. Hence the name of the theory.

A quantum field theory seeks to explain certain fundamental experimental observations—the existence of antiparticles, the spin-statistics relation, the CPT symmetry—as well as predict the results of any given experiment, like the cross-section for the Compton scattering, or the value of the anomalous magnetic moment of the electron.

There are two popular approaches to deal with quantum fields. One is the path integral formulation, where elementary particles have the property of being able to propogate simultaneously along all possible trajectories with certain apmplitudes. In the other approach—called canonical quantization—elementary particles are field quanta: necessarily chunked ripples in the field.

In the end, the two formulations proved to be equivalent.

We shall pursue the second approach here and build the quantum field theory in the canonical way: as a classical Lagrangian field theory with the subsequent canonical quantization[1].

Fundamental principles

Quantum field theory is built on several fundamental principles. A principle is a physical law of more general—typically universal— applicability usually formulated as a simple and succinct statement.

Principle of relativity

The principle of relativity is the requirement that the laws of physics have the same form in all admissible frames of reference.

In the absence of gravitation one can choose to admit only inertial frames of reference. The laws of physics take particularly simple form in inertial frames.

[1] sometimes historically called “second quantization”.

Principle of locality

The principle of locality states that an object can only be influenced by its immediate surroundings.

From this principle follows the finite speed of information transmission.

Principle of covariance

The principle of covariance emphasizes formulation of physical laws using only those physical quantities the measurements of which the observers in different frames of reference could unambiguously correlate.

Mathematically speaking, the physical quantities must transform covariantly, that is, under a certain representation of the group of coordinate transformations between admissible frames of reference of the physical theory. This group of coordinate transformations is referred to as the covariance group of the theory.

The principle of covariance does not require invariance of physical laws under the group of admissible transformations although in most cases the equations are actually invariant. Only in the theory of weak interactions the equations are not invariant under reflections (but are, of course, still covariant).

In canonical quantum field theory the admissible frames of reference are the inertial frames of special relativity. The transformations between frames are the velocity boosts, rotations, translations, and reflections. Altogether they form the Poincaré group of coordinate transformations. Boosts and rotations together make up the Lorentz group.

The convariant quantities are four-scalars, four-vectors etc. of the Minkowski space of special relativity (and also more complicated objects like bispinors and others which we shall discuss later).

Covariant vectors of special relativity

Four-coordinates

An *event* in an inertial frame can be specified with four coordinates $\{t, \mathbf{r}\}$, where t is the time of the event, and $\mathbf{r} \equiv \{x, y, z\}$ are the three Euclidean spatial coordinates.

The coordinates of the same event in different inertial frames are connected by a linear transformation from the Poincaré group. Rotations, translations and reflections do not couple time and spatial coordinates, but velocity boosts do. Therefore in special relativity time and spatial coordinates are inseparable components of one and the same object. It is only in the non-relativistic limit that time separates from space.

The Lorentz transformation of coordinates under a velocity boost v along the x-axis is given as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} , \quad (1)$$

where primes denote coordinates in the boosted frame, $\gamma \equiv (1 - v^2/c^2)^{-1/2}$, and c is the speed of light in vacuum[2].

The four-coordinates $\{t, \mathbf{r}\}$ are customarily denoted as x^a, where $a = 0 \ldots 3$, such that

$$x^0 = t,\ x^1 = x,\ x^2 = y,\ x^3 = z\ . \quad (2)$$

The Lorentz transformation (1) can then be conveniently written as[3]

$$x'^a = \Lambda^a_b x^b\ . \quad (3)$$

where Λ^a_b is the 4×4 transformation matrix in equation (1).

Invariant

A direct calculation shows that velocity boosts (together with rotations, translations, and reflections) conserve the following form,

$$s^2 = t^2 - \mathbf{r}^2\ , \quad (4)$$

which is then called the invariant *interval.*

In its infinitesimal incarnation,

$$ds^2 = dt^2 - d\mathbf{r}^2\ , \quad (5)$$

the form determines the geometry of time-space and is called *metric.* The space with metric (5) is called Minkowski space.

Dual coordinates

The invariant (4) can be conveniently written as

$$t^2 - \mathbf{r}^2 \equiv x_a x^a\ , \quad (6)$$

where x_a are often called *dual coordinates* and are defined as

$$x_a \equiv \{t, -\mathbf{r}\} = g_{ab} x^b\ , \quad (7)$$

where the diagonal tensor g_{ab} with the main diagonal $\{1, -1, -1, -1\}$ is the metric tensor of the Minkowski space of special relativity.

Under Lorentz transformations the dual coordinates apparently transform with the Lorentz matrix where v is substituted by $-v$, which is actually the inverse Lorentz matrix (prove it),

$$x'_a = (\Lambda^{-1})^b_a x_b\ , \quad (8)$$

[2]in the following the notation with $\hbar = c = 1$ shall be mostly used.

[3]note the "implicit summation" notation, $\Lambda^a_b x^b \equiv \sum_{b=0}^{3} \Lambda^a_b x^b$.

Four-gradient

The partial derivatives of a scalar with respect to four-coordinates,

$$\partial_a \equiv \frac{\partial}{\partial x^a} = \left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} \equiv \left\{\frac{\partial}{\partial t}, \nabla\right\}\ , \quad (9)$$

apparently transform like dual coordinates, that is, via the inverse Lorentz matrix,

$$\frac{\partial}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a}\frac{\partial}{\partial x^b} = (\Lambda^{-1})^b_a \frac{\partial}{\partial x^b}\ . \quad (10)$$

Covariant vectors and tensors

A *contra-variant four-vector* is a set of four objects, $A^a = \{A^0, \mathbf{A}\}$, which transform from one inertial frame to another in the same way as coordinates in (3),

$$A'^a = \Lambda^a_b A^b\ . \quad (11)$$

A *co-variant four-vector* is a set of four objects, A_a, which transform from one inertial frame to another in the same way as partial derivatives in (10),

$$A'_a = (\Lambda^{-1})^b_a A_b\ . \quad (12)$$

A *covariant tensor* F^{ab} is a set of 4×4 objects which transform between inertial frames as a product of two 4-vectors,

$$F'^{ab} = \Lambda^a_c \Lambda^b_d F^{cd}\ . \quad (13)$$

There exist other covariant objects in special relativity, like bispinors, which cannot be built out of 4-vectors. They will be discussed later.