Massive spin-1 field

The lowest-dimension representation of the full Lorentz group, containing spin-1, is $(\frac{1}{2}, \frac{1}{2})$. The matrices of this representations transform four-vectors $\varphi^a = \{\varphi^0, \vec{\varphi}\}$.

However, a four-vector also contains a rotation scalar, φ^0, that is a spin-0 part. This redundant component has to be excluded by imposing some additional condition, for example, the (covariant) Lorenz condition[1],

$$\partial_a \varphi^a = 0 \; . \tag{1}$$

For a plane wave $\varphi^a = \epsilon^a e^{-ipx}$, where ϵ^a is a four-vector and $p_a p^a = m^2$, the Lorenz condition gives $\epsilon p = 0$. In the rest frame ($\mathbf{p} = 0$) the latter leads to $\epsilon^0 = 0$ indicating that only three components of the vector $\vec{\epsilon}$ are independent, which is consistent with the concept of a spin-1 field.

Lagrangian

A suitable Lagrangian (that is, real bilinear form of fields and field derivatives) is

$$\mathcal{L} = -\partial_a \varphi_b^* \partial^a \varphi^b + m^2 \varphi_b^* \varphi^b \; . \tag{2}$$

This Lagrangian is simply a sum of Klein-Gordon Lagrangians for each field component ϕ^b. Therefore most of the results from the scalar fields immediately apply for spin-1 fields. In particular, the spin-1 field must be a bosonic field with bosonic annihilation/generation operators.

Euler-Lagrange equation

The general form of the Euler-Lagrange equation,

$$\partial_a \frac{\partial \mathcal{L}}{\partial(\partial_a \varphi_b^*)} = \frac{\partial \mathcal{L}}{\partial(\varphi_b^*)} \; , \tag{3}$$

leads to the Klein-Gordon equation for every component φ^b,

$$\left(\partial_a \partial^a + m^2\right) \varphi^b = 0 \; . \tag{4}$$

Normal modes

$$\varphi = \sum_{\mathbf{k}\lambda} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}\lambda} \epsilon_\lambda e^{-ikx} + b^+_{\mathbf{k}\lambda} \epsilon^*_\lambda e^{+ikx} \right) \; , \tag{5}$$

where the spin functions ϵ_λ are chosen in the rest frame (where $(\epsilon_\lambda)^0 = 0$) as eigenfunction of the I_3 generator, $I_3 \epsilon_\lambda = \lambda \epsilon_\lambda$, $\lambda = 1, 0, -1$. They are normalized as $\epsilon^\dagger_\lambda \epsilon_{\lambda'} = \delta_{\lambda\lambda'}$.

The generation/annihilation operators a and b satisfy bosonic commutation relations,

$$a_{\mathbf{k}\lambda} a^\dagger_{\mathbf{k}\lambda} - a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\lambda\lambda'} \; . \tag{6}$$

[1] named after Danish physicist Ludvig Lorenz.

Electromagnetic field

Electromagnetic field is a real massless spin-1 field. It is described by a real four-vector potential A^a. The Lagrangian of the electromagnetic field in Gaussian units is given as

$$\mathcal{L} = -\frac{1}{8\pi} \partial_a A^b \partial^a A_b \; . \tag{7}$$

Equivalently, the Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{16\pi} F^{ab} F_{ab}; \; , \tag{8}$$

where F^{ab} is the electromagnetic tensor,

$$F^{ab} = \partial^a A^b - \partial^b A^a \; . \tag{9}$$

Gauge invariance

The absence of the mass term in the Lagrangian leads to an important symmetry, called *gauge symmetry*: the Lagrangian is invariant under the so-called *gauge transformation*,

$$A^a \to A^a + \partial^a \phi \; , \tag{10}$$

where ϕ is an arbitrary scalar function of coordinates. Indeed the electromagnetic tensor (9) and therefore the Lagrangian (8) are apparently invariant under the gauge transformation (10).

The Lorenz condition limits the class of arbitrary function to harmonic functions

$$\partial^a \partial_a \phi = 0 \; . \tag{11}$$

Normal modes

The Euler-Lagrange equation for the components of the four-potential is the zero-mass Klein-Gordon equation,

$$\partial^a \partial_a A^b = 0 \; . \tag{12}$$

Let us look for the solutions in the form of a plane wave,

$$A(k) = \sqrt{\frac{2\pi}{\omega}} e(k) e^{-ikx} \; , \tag{13}$$

where $k = \{\omega, \mathbf{k}\}$, $\omega = |\mathbf{k}|$, $k^2 = 0$ (since photon mass is zero), and $e(k)$ is a four-vector.

From the Lorenz condition,

$$ke \equiv \omega e^0 - \vec{k}\vec{e} = 0 \; , \tag{14}$$

it follows that $\vec{e} \neq 0$, that is the amplitude $e(k)$ is not time-like.

A gauge transformation with $\phi = \sqrt{\frac{2\pi}{\omega}} i f e^{-ikx}$, where f is a scalar, leads to a transformation of the amplitude

$$e_a \to e_a + f k_a \; . \tag{15}$$

The scalar f can always be chosen such that in a certain frame

$$e = \{0, \mathbf{e}\} \ , \ \ \mathbf{ke} = 0; \tag{16}$$

and

$$A_0 = 0 \ , \ \ \nabla \mathbf{A} = 0 \ . \tag{17}$$

The gauge (17) is usually referred to as *Coulomb gauge*, *transverse gauge*, or *radiation gauge*. In this gauge the electromagnetic field can be represented as

$$\mathbf{A} = \sum_{\mathbf{k}\lambda} \sqrt{\frac{2\pi}{\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}\lambda} \mathbf{e}_\lambda e^{-ikx} + a^\dagger_{\mathbf{k}\lambda} \mathbf{e}^*_\lambda e^{ikx} \right) \ , \tag{18}$$

where $\lambda = 1, 2$ are the two orthogonal polarizations,

$$\mathbf{e}_\lambda \mathbf{e}_{\lambda'} = \delta_{\lambda\lambda'} \ , \ \ \mathbf{ek} = 0 \ . \tag{19}$$

Since A^a is real, there are no "anti"-particle operators in the normal mode expansion (18) – real fields have only one type of particles.

The generation/annihilation operators $a_{\mathbf{k}\lambda}$, $a^\dagger_{\mathbf{k}\lambda}$ satisfy the bosonic commutation relations,

$$a_{\mathbf{k}\lambda} a^\dagger_{\mathbf{k}\lambda} - a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} = \delta_{\mathbf{kk}'} \delta_{\lambda\lambda'} \ . \tag{20}$$

Spin-statistics theorem

The spin-statistics theorem states that relativistically covariant canonical quantum field theory with positive definite energy density predicts that fields with integer spin are necessarily bosonic, and fields with half-integer spin – necessarily fermionic.

Exercises

1. Show that the Lagrangians

$$\mathcal{L} = -\frac{1}{8\pi} \partial_a A^b \partial^a A_b$$

and

$$\mathcal{L} = -\frac{1}{16\pi} F^{ab} F_{ab}$$

are equivalent under the Lorenz condition $\partial_a A^a = 0$. Hint: the difference is a full derivative, which does not contribute to the variation of the action.

2. Show that the Lagrangian

$$\mathcal{L} = -\frac{1}{8\pi} \partial_a A^b \partial^a A_b - j^a A_a$$

is gauge invariant if j^a is a conserved current ($\partial_a j^a = 0$).

3. Derive the Maxwell equation with sources from the Lagrangian

$$\mathcal{L} = -\frac{1}{8\pi} \partial_a A^b \partial^a A_b - j^a A_a$$

4. Derive the expression $q\left(\mathbf{E} + \mathbf{v} \times \mathbf{H}\right)$ for the Lorentz force (the electromagnetic force acting on a charged particle) from the action

$$S = -m \int ds - q \int dx^a A_a \ ,$$

where the integral is take along the trajectory of the particle, $ds^2 = dt^2 - d\mathbf{r}^2$, m and q are the mass and the charge of the particle, A_a is the electromagnetic field at the point where the particle is located.