Canonical quantization of spin-$\frac{1}{2}$ field

Euler-Lagrange equation, current, energy

The simplest covariant Lagrangian for a spin-$\frac{1}{2}$ field is (the real part of)

$$\mathcal{L} = i\bar{\psi}\gamma^a\partial_a\psi - m\bar{\psi}\psi \; . \tag{1}$$

The corresponding Euler-Lagrange equation is the Dirac equation,

$$(i\gamma^a\partial_a - m)\psi = 0 \; . \tag{2}$$

The Noether's conserved current is

$$j^a = i\left(\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial_a\bar{\psi})} - \frac{\partial\mathcal{L}}{\partial(\partial_a\psi)}\psi\right) = \bar{\psi}\gamma^a\psi \; , \tag{3}$$

and the canonical energy density is

$$T^0_0 = -i\bar{\psi}\vec{\gamma}\vec{\partial}\psi + m\bar{\psi}\psi \; . \tag{4}$$

Normal modes

Let us find the complete basis of solutions to the Dirac equation in the form of plane waves,

$$\psi(x) = \begin{pmatrix}\phi\\ \chi\end{pmatrix} e^{-ipx} \equiv \begin{pmatrix}\phi\\ \chi\end{pmatrix} e^{i\vec{p}\vec{r}-iEt} \; , \tag{5}$$

where $e^{-ipx} \equiv e^{i\vec{p}\vec{r}-iEt}$ and the spinors ϕ and χ are no longer functions of time and coordinates. Substituting the plane-wave ansatz (5) into the Dirac equation (2) gives (in the Dirac basis)

$$\begin{bmatrix} E-m & -\vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & -(E+m)\end{bmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = 0 \; . \tag{6}$$

The homogeneous system of linear algebraic equations has a non-trivial solution only when the determinant of the matrix is zero – this gives[1] the relativistic relation between E and $\vec{p}$,

$$E^2 = m^2 + \vec{p}^{\,2} \; . \tag{7}$$

Again there are both positive- and negative-frequency solutions

$$u_{\vec{p}}\, e^{-ipx} \; , \text{ and } \; v_{\vec{p}}\, e^{+ipx} \; , \tag{8}$$

where $p^a = \{E_{\vec{p}}\,, \vec{p}\}$, $\vec{p}$ is an arbitrary wave-vector, and $E_{\vec{p}} = +\sqrt{m^2+\vec{p}^{\,2}}$ is the positive square root.

From the Dirac equation, the positive- and negative-frequency bispinors $u_{\vec{p}}$ and $v_{\vec{p}}$ are

$$u_{\vec{p}} = \begin{pmatrix}\phi\\ \frac{\vec{\sigma}\vec{p}}{E_{\vec{p}}+m}\phi\end{pmatrix} , \quad v_{\vec{p}} = \begin{pmatrix}\frac{\vec{\sigma}\vec{p}}{E_{\vec{p}}+m}\chi \\ \chi\end{pmatrix} . \tag{9}$$

[1] $(\vec{\sigma}\vec{p})^2 = \vec{p}^2$

where ϕ and χ are arbitrary spinors.

There are two linearly independent spinors, which can be chosen for the complete basis. For example, one can choose eigen-spinors of the $\frac{1}{2}\sigma_3$ operator in the frame where $\vec{p}$=0,

$$\frac{1}{2}\sigma_3\phi_\lambda = \lambda\phi_\lambda \; , \tag{10}$$

and similarly for the spinor χ.

Charge and normalization The charge density for the spin-$\frac{1}{2}$ field is

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi \; . \tag{11}$$

The charge $Q = \int_{V=1} d^3x j^0$ of the normal modes is then given as

$$Q\left(u_{\vec{p}\lambda}e^{-ipx}\right) = u^\dagger_{\vec{p}\lambda}u_{\vec{p}\lambda} = \phi^\dagger_\lambda\phi_\lambda\frac{2E_{\vec{p}}}{E_{\vec{p}}+m} \; , \tag{12}$$

$$Q\left(v_{\vec{p}\lambda}e^{+ipx}\right) = v^\dagger_{\vec{p}\lambda}v_{\vec{p}\lambda} = \chi^\dagger_\lambda\chi_\lambda\frac{2E_{\vec{p}}}{E_{\vec{p}}+m} \; . \tag{13}$$

Unlike the scalar field the spin-$\frac{1}{2}$ field seemingly gives positive charges for both positive- and negative-frequency solutions.

Unit charge condition for a plane-wave leads to the normalizations

$$u^\dagger_\lambda u_{\lambda'} = v^\dagger_\lambda v_{\lambda'} = \delta_{\lambda\lambda'} \; , \tag{14}$$

which are obtained by choosing

$$\phi^\dagger_\lambda\phi_{\lambda'} = \chi^\dagger_\lambda\chi_{\lambda'} = \frac{E_{\vec{p}}+m}{2E_{\vec{p}}}\delta_{\lambda\lambda'} \; . \tag{15}$$

With this normalization

$$\bar{u}_\lambda u_{\lambda'} = -\bar{v}_\lambda v_{\lambda'} = \frac{m}{E_{\vec{p}}}\delta_{\lambda\lambda'} \; . \tag{16}$$

Energy From (4) follows that the positive-energy solution has positive energy,

$$E\left(ue^{-ipx}\right) = \bar{u}\vec{\gamma}\vec{p}u + m\bar{u}u = E_{\vec{p}} \; , \tag{17}$$

while the negative-energy solution seemingly has negative energy,

$$E\left(ve^{+ipx}\right) = \bar{v}\vec{\gamma}(-\vec{p})v + m\bar{v}v = -E_{\vec{p}} \; . \tag{18}$$

Normal mode representation

An arbitrary solution to the Dirac equation can be represented as a linear combination of normal modes,

$$\psi = \sum_{\vec{p}\lambda}\left(a_{\vec{p}\lambda}u_{\vec{p}\lambda}e^{-ipx} + b^\dagger_{\vec{p}\lambda}v_{\vec{p}\lambda}e^{-ipx}\right) \; . \tag{19}$$

The charge of the field is then given as

$$Q = \sum_{\vec{p}\lambda} \left(a^\dagger_{\vec{p}\lambda} a_{\vec{p}\lambda} + b_{\vec{p}\lambda} b^\dagger_{\vec{p}\lambda} \right) , \quad (20)$$

and the Hamiltonian,

$$H = \sum_{\vec{p}\lambda} E_{\vec{p}} \left(a^\dagger_{\vec{p}\lambda} a_{\vec{p}\lambda} - b_{\vec{p}\lambda} b^\dagger_{\vec{p}\lambda} \right) . \quad (21)$$

Generation/annihilation operators

The only way to make sense of the charge and Hamiltonian is to postulate *anti-commutation* relation for the spin-$\frac{1}{2}$ operators,

$$a_{\vec{p}\lambda} a^\dagger_{\vec{p}\,'\lambda'} + a^\dagger_{\vec{p}\,'\lambda'} a_{\vec{p}\lambda} = \delta_{\vec{p}\vec{p}\,'} \delta_{\lambda\lambda'} , \quad (22)$$

$$b_{\vec{p}\lambda} b^\dagger_{\vec{p}\,'\lambda'} + b^\dagger_{\vec{p}\,'\lambda'} b_{\vec{p}\lambda} = \delta_{\vec{p}\vec{p}\,'} \delta_{\lambda\lambda'} . \quad (23)$$

With this postulates the charge and the energy take the form as prescribed by the experiment,

$$Q = \sum_{\vec{p}\lambda} \left(n_{\vec{p}\lambda} - \bar{n}_{\vec{p}\lambda} \right) , \quad (24)$$

$$E = \sum_{\vec{p}\lambda} E_{\vec{p}} \left(n_{\vec{p}\lambda} + \bar{n}_{\vec{p}\lambda} \right) , \quad (25)$$

where $n_{\vec{p}\lambda} = a^\dagger_{\vec{p}\,'\lambda'} a_{\vec{p}\lambda}$ and $\bar{n}_{\vec{p}\lambda} = b^\dagger_{\vec{p}\,'\lambda'} b_{\vec{p}\lambda}$ are the number-of-particle and number-of-anti-particle operators with eigenvalues 0 and 1.

Thus in canonical quantum field theory spin-$\frac{1}{2}$ particles are necessarily fermions.

Exercises

1. Show that for a classical particle with the action
$$S = \int dt L(q, \dot{q})$$
   (a) the Euler-Lagrange equation is
$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} .$$
   (b) the (translation invariance) momentum is
$$p = \frac{\partial L}{\partial \dot{q}} .$$
   (c) the (time invariance) energy is
$$E = \frac{\partial L}{\partial \dot{q}} \dot{q} - L .$$

2. Show that for a classical particle with mass m the action
$$S = -m \int ds ,$$
where $ds^2 = dt^2 - d\vec{r}^{\,2}$ leads to the relativistic relation between energy and momentum,
$$E^2 = m^2 + \vec{p}^{\,2} .$$
Hints:
   (a) Show that the Lagrangian is
$$L = -m\sqrt{1 - \vec{v}^{\,2}}.$$
   (b) Show that the momentum is
$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - \vec{v}^{\,2}}}.$$
   (c) Show that the energy is
$$E = \frac{m}{\sqrt{1 - \vec{v}^{\,2}}}.$$

3. Calculate the current and the momentum of the positive- and negative-frequency solutions of the Dirac equation.

4. Find the projection operators $\mathcal{P}_\pm$ on the positive and negative frequency solutions of the Dirac equation.

5. The spinors $\phi \in (\frac{1}{2}, 0)$ and $\chi \in (0, \frac{1}{2})$ are often called “left” and “right”. Find out why. Hint: find the projection of the spin on the velocity vector as function of the velocity with which the spinor moves relative to the observer: consider a state with equal x-projections of the spin and then boost it along the x-axis. What happens if the spinor has zero-mass and is thus doomed to forever move with the speed of light?

6. Show that $\frac{d^3p}{2E_{\vec{p}}}$ is invariant.

   Hint: consider $d^4p\delta(p^2 - m^2)$.