Spin-$\frac{1}{2}$ field

Bispinors

The lowest irreducible representation of the full Lorentz group $O(1,3)$, containing spin-$\frac{1}{2}$, is

$$(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\,. \tag{1}$$

The matrices from this representation transform four-component objects called *bispinors* or Dirac spinors,

$$\psi=\left(\begin{array}{c}\phi\\ \chi\end{array}\right)\,, \tag{2}$$

where the two-component objects $\phi\sim(\frac{1}{2},0)$ and $\chi\sim(0,\frac{1}{2})$ are the (right and left) *Weyl spinors.* Their infinitesimal transformation matrices are

$$\mathsf{t}_{(\frac{1}{2},0)}=1+i\tfrac{1}{2}\vec{\sigma}d\vec{w}+i0d\vec{w}^{\star}\,, \tag{3}$$

and

$$\mathsf{t}_{(0,\frac{1}{2})}=1+i0d\vec{w}+i\tfrac{1}{2}\vec{\sigma}d\vec{w}^{*}\,. \tag{4}$$

Apparently,

$$\mathsf{t}_{(0,\frac{1}{2})}=\left(\mathsf{t}^{\dagger}_{(\frac{1}{2},0)}\right)^{-1}\,, \tag{5}$$

and therefore under a Lorentz transformation Λ

$$\phi\xrightarrow{\Lambda}\mathsf{t}\phi\;,\;\chi\xrightarrow{\Lambda}\mathsf{t}^{\dagger\,-1}\chi\,, \tag{6}$$

where $\mathsf{t}\equiv\mathsf{t}_{(\frac{1}{2},0)}$.

Under the parity transformation, $\mathcal{P}$, the Weyl spinors transform into each other,

$$\phi\overset{\mathcal{P}}{\longleftrightarrow}\chi\,. \tag{7}$$

Bilinear forms of bispinors

The Lagrangian for the filed must be a real scalar, bilinear in the field. That is, it has to be built out of $\psi^{+}\otimes\psi$ which transforms under the representation

$$\left((\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\right)^{*}\otimes\left((\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\right)\,. \tag{8}$$

According to the general rule this direct product reduces to a direct sum of the following representations,

$$(0,0)\;;\;(0,1)\oplus(1,0)\;;\;(\tfrac{1}{2},\tfrac{1}{2})\,, \tag{9}$$

where the first is scalar, the second – antisymmetric tensor, and the third – four-vector.

Scalars

Using the transformation laws (6) and (7) it is easy to show that the (real) scalar is the combination

$$\bar{\psi}\psi\equiv\phi^{+}\chi+\chi^{+}\phi\,, \tag{10}$$

where $\bar{\psi}\equiv\psi^{\dagger}\gamma^{0}$ and the block-matrix γ^{0} is given as

$$\gamma^{0}=\left[\begin{array}{cc}0&\mathbf{1}\\ \mathbf{1}&0\end{array}\right]\,. \tag{11}$$

The combination

$$i\bar{\psi}\gamma^{5}\psi\equiv-i(\phi^{+}\chi-\chi^{+}\phi)\,, \tag{12}$$

where

$$\gamma^{5}=\left[\begin{array}{cc}\mathbf{1}&0\\ 0&-\mathbf{1}\end{array}\right] \tag{13}$$

is a (real) *pseudo-scalar*, that is, it changes sign under parity transformation.

Vectors

Since $\phi=(\frac{1}{2},0)$ and $\phi^{*}=(0,\frac{1}{2})$, their direct product is a four-vector,

$$\phi^{*}\otimes\phi=(\tfrac{1}{2},\tfrac{1}{2})\,. \tag{14}$$

Apparently, this four-vector is

$$\{\phi^{+}\phi,\phi^{+}\vec{\sigma}\phi\}\,. \tag{15}$$

Analogously,

$$\{\chi^{+}\chi,\chi^{+}\vec{\sigma}\chi\} \tag{16}$$

is also a four-vector.

Now we can build a *polar vector*,

$$\bar{\psi}\gamma^{a}\psi\equiv\{\phi^{+}\phi+\chi^{+}\chi,\phi^{+}\vec{\sigma}\phi-\chi^{+}\vec{\sigma}\chi\}\,, \tag{17}$$

and an *axial-* or *pseudo-vector*,

$$i\bar{\psi}\gamma^{a}\gamma^{5}\psi\equiv i\{\phi^{+}\phi-\chi^{+}\chi,\phi^{+}\vec{\sigma}\phi+\chi^{+}\vec{\sigma}\chi\}\,, \tag{18}$$

where $\gamma^{a}=\{\gamma^{0},\vec{\gamma}\}$ and

$$\vec{\gamma}=\left[\begin{array}{cc}0&-\vec{\sigma}\\ \vec{\sigma}&0\end{array}\right]\,. \tag{19}$$

Antisymmetric tensor

The representation $(0,1)$ can be made of

$$\phi^{*}\otimes\chi\sim(0,\tfrac{1}{2})\otimes(0,\tfrac{1}{2})\ni(0,1)\,, \tag{20}$$

and, correspondingly, $(1,0)$ out of

$$\chi^{*}\otimes\phi\sim(\tfrac{1}{2},0)\otimes(\tfrac{1}{2},0)\ni(1,0)\,. \tag{21}$$

Under rotations the objects $(0,1)$ and $(1,0)$ transform as pure vectors. Apparently they are

$$\phi^{+}\vec{\sigma}\chi\sim(0,1) \tag{22}$$

and

$$\chi^{+}\vec{\sigma}\phi \sim (1,0)\ . \qquad (23)$$

Under reflections these two objects transform into each other. Therefore their sum and difference is a six-component objects that transforms through itself under the full Lorentz group $O(1,3)$.

Using *gamma*-matrices this tensor is usually written as

$$\tfrac{1}{2}\bar{\psi}(\gamma^a\gamma^b - \gamma^b\gamma^a)\psi\ . \qquad (24)$$

Resume

The direct product $\psi^* \otimes \psi$ of two bispinors can be reduced into five irreducible covariant objects:

1. scalar, $\bar{\psi}\psi$;
2. pseudo-scalar, $i\bar{\psi}\gamma_5\psi$;
3. vector, $\bar{\psi}\gamma^a\psi$;
4. pseudo-vector, $i\bar{\psi}\gamma^a\gamma_5\psi$;
5. antisymmetric tensor, $\frac{1}{2}\bar{\psi}(\gamma^a\gamma^b - \gamma^b\gamma^a)\psi$,

where $\bar{\psi} \equiv \psi^\dagger\gamma_0$ and the block-matrices γ^a and γ^5 are called *gamma-matrices* or Dirac matrices.

Gamma matrices

The bispinors and gamma-matrices in the previous section are given in the so-called Weyl or *chiral* basis. A similarity transformation

$$\begin{array}{rcl} \psi & \rightarrow & S\psi \\ \gamma^a & \rightarrow & S\gamma^a S^{-1} \end{array} \qquad (25)$$

with the matrix

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad (26)$$

changes Weyl basis into Dirac or standard basis, where the γ-matrices are

$$\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (27)$$

$$\vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix} \qquad (28)$$

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\ . \qquad (29)$$

The γ-matrices satisfy the anti-commutation relation

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2g^{ab}\ , \qquad (30)$$

where g^{ab} is the Minkowski metric tensor. The matrix γ_5 anti-commutes with γ^a,

$$\{\gamma^a, \gamma_5\} = 0\ . \qquad (31)$$

Lagrangian of spin-$\frac{1}{2}$ field

The Lagrangian must be a real scalar, bilinear in ψ and $\partial_a\psi$. A suitable form is

$$\mathcal{L} = \frac{i}{2}\left(\bar{\psi}\gamma^a\partial_a\psi - \partial_a\bar{\psi}\gamma^a\psi\right) - m\bar{\psi}\psi\ . \qquad (32)$$

Dirac equation

The general expression for the Euler-Lagrange equation,

$$\partial_a\frac{\partial\mathcal{L}}{\partial(\partial_a\bar{\psi})} = \frac{\partial\mathcal{L}}{\partial\bar{\psi}}\ , \qquad (33)$$

with the Lagrangian (32) leads to the so-called *Dirac equation*,

$$(i\gamma^a\partial_a - m)\psi = 0\ . \qquad (34)$$

Energy-momentum tensor and conserved current

Again, using the general expressions, the energy-momentum tensor and the conserved current are directly obtained from the Lagrangian,

$$T_0^0 = -i\bar{\psi}\vec{\gamma}\vec{\partial}\psi + m\bar{\psi}\psi\ , \qquad (35)$$

$$j^a = \bar{\psi}\gamma^a\psi\ . \qquad (36)$$

Exercises

1. Argue that the 2×2 representation of the Lie algebra[1] of the rotation group is the group of special unitary 2×2 matrices (called $SU(2)$). In other words, argue that $SU(2)$ group has the same Lie algebra, as the rotation group $SO(3)$.
2. Show that after a rotation by 2π a spinor[2] changes sign.
3. Show that the Lagrangian
$$\mathcal{L} = \tfrac{1}{2}\left(\bar{\psi}\gamma^a\partial_a\psi + \partial_a\bar{\psi}\gamma^a\psi\right) - m\bar{\psi}\psi$$
does not lead to a good Euler-Lagrange equation.
4. Show by direct calculation that the current $\bar{\psi}\gamma^a\psi$ conserves if ψ is a solution of the Dirac equation.
5. Calculate the energy-momentum tensor for the spin-1/2 field.

[1] in physics, a representation of a Lie algebra is a group of matrices whose generators have the given Lie algebra

[2] here 'spinor' is an object transformed by a 2×2 representation of the rotation group.

6. Show that if bispinor ψ is a solution to the Dirac equation, then its components satisfy the Klein-Gordon equation[3].

[3] Hint: multiply the Dirac equation by $(i\gamma^b \partial_b + m)$ from the left.