Finite rotation matrix

The generators can recover not only infinitesimal, $g(d\alpha) = 1 + iId\alpha$, but (in principle) also finite elements of the group (with certain caveats). It is especially easy with *additive parameters*, where $g(\alpha + d\alpha) = g(\alpha)g(d\alpha)$. Indeed, in this case

$$g(\alpha + d\alpha) = g(d\alpha)g(\alpha) = (1 + iId\alpha)g(\alpha) , \quad (1)$$

and thus the group element $g(\alpha)$ satisfies the differential equation

$$\frac{\partial g(\alpha)}{\partial \alpha} = iIg(\alpha) , \quad (2)$$

with the initial condition $g(0) = 1$. One can check by direct substitution that the solution to this equation is exponential function, understood as an infinite Taylor series

$$g(\alpha) = e^{iI\alpha} \equiv \sum_{n=0}^{\infty} \frac{(iI\alpha)^n}{n!} . \quad (3)$$

For example, for the rotation around a given axis $\vec{n}$ the rotation angle θ is an additive parameter of the rotation matrix $R(\vec{n}, \theta)$,

$$\begin{aligned} R(\vec{n}, \theta + d\theta) &= R(\vec{n}, \theta)R(\vec{n}, d\theta) \\ &= (1 + i\vec{I}\vec{n}d\theta)R(\vec{n}, \theta) , \end{aligned} \quad (4)$$

and thus the rotation matrix satisfies the differential equation

$$\frac{dR}{d\theta} = i\vec{I}\vec{n}R , \quad (5)$$

with the boundary condition $R(\vec{n}, 0) = 1$. The solution is given by the Taylor series,

$$R(\vec{n}, \theta) = e^{i\vec{I}\vec{n}\theta} \equiv 1 + i\vec{I}\vec{n}\theta + \frac{(i\vec{I}\vec{n}\theta)^2}{2!} + \ldots . \quad (6)$$

In practice for a finite-dimension representation there is only a finite number of terms in the series.

Direct product of two representations

In the field theory we often have to work with different types of products of covariant quantities, like $\partial_a j^a$ or $\partial_a A^b$. We need to know how such products transform.

For simplicity we shall only talk about direct products and direct sums of matrices, although the concepts are also defined for abstract groups.

A *direct* product $g \otimes h$ of two matrices g and h is a matrix made of all possible (and suitable arranged) pairwise products of matrix elements of g and h,

$$g \otimes h = \begin{bmatrix} g_{11}h_{11} & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & g_{n_g n_g} h_{n_h n_h} \end{bmatrix} , \quad (7)$$

where n_g and n_h are the sizes of g and h. The size of the direct product is $n_{g\otimes h} = n_g n_h$.

If the matrices g and h act on column-vectors v_g and v_h of sizes n_g and n_h, then the matrices $g \otimes h$ act on a column-vector $v_g \otimes v_h$ of size $n_g n_h$ consisting of all pairwise products of the elements of vectors v_g and v_h,

$$v_g \otimes v_h = \begin{bmatrix} (v_g)_1(v_h)_1 \\ \ldots \\ (v_g)_{n_g}(v_h)_{n_h} \end{bmatrix} . \quad (8)$$

A *direct sum* $g \oplus h$ of two matrices g and h is a block-diagonal matrix made of matrices g and h,

$$g \oplus h = \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} . \quad (9)$$

The size of the direct sum is $n_{g\oplus h} = n_g + n_h$.

If the matrices g and h act on column-vectors v_g and v_h of sizes n_g and n_h, then the matrices $g \oplus h$ act on a column-vector $v_g \oplus v_h$ of size $n_g + n_h$ which is made of elements of both vectors v_g and v_h,

$$v_g \oplus v_h = \begin{bmatrix} v_g \\ v_h \end{bmatrix} . \quad (10)$$

Suppose we have two (different) representations, $G = \{g\}$ and $H = \{h\}$, of some Lie group $\{\Lambda\}$, with the corresponding infinitesimal elements[1]

$$g = 1_G + i\vec{I}_G\vec{\alpha} , \quad (11)$$

$$h = 1_H + i\vec{I}_H\vec{\alpha} , \quad (12)$$

where $\vec{\alpha}$ are the parameters of the group $\{\Lambda\}$.

It is easy to show that the generators $I_{G\otimes H}$ of a direct product $G \otimes H$ of two representations G and H of the same group is a (Kronecker) sum of the corresponding generators I_G and I_H,

$$\vec{I}_{G\otimes H} = \vec{I}_G \otimes 1_H + 1_G \otimes \vec{I}_H . \quad (13)$$

Indeed,

$$\begin{aligned} g \otimes h &= \left(1_G + i\vec{I}_G\vec{\alpha}\right) \otimes \left(1_H + i\vec{I}_H\vec{\alpha}\right) \\ &= 1_G \otimes 1_H + i\left(\vec{I}_G \otimes 1_H + 1_G \otimes \vec{I}_H\right)\vec{\alpha} \\ &= 1 + i\vec{I}_{G\otimes H}\vec{\alpha} , \quad (14) \\ &\Rightarrow \vec{I}_{G\otimes H} = \vec{I}_G \otimes 1_H + 1_G \otimes \vec{I}_H . \quad (15) \end{aligned}$$

Clebsch-Gordan theorem

The direct product $(j_1) \otimes (j_2)$ of two irreducible representations of the rotation group is a reducible

[1] $\vec{I}\vec{\alpha} \equiv \sum_k I_k \alpha_k$

representation which can be reduced into a direct sum of irreducible representations,

$$(j_1) \otimes (j_2) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \oplus (j) \, . \tag{16}$$

Example: a direct product $\vec{a} \otimes \vec{b}$ of two vectors, $(1) \otimes (1)$, reduces to a direct sum of a scalar ($j = 0$), an antisymmetric tensor ($j = 1$), and a symmetric tensor with zero trace ($j = 2$),

$$\vec{a} \otimes \vec{b} = \left(\vec{a}\vec{b}\right) \oplus \left(\vec{a} \times \vec{b}\right) \oplus \left(a_i b_j + a_j b_i - \frac{2}{3}(\vec{a}\vec{b})\delta_{ij}\right) . \tag{17}$$

Irreducible representations of the Lorentz group

With the complex parameterization

$$d\vec{w} = \vec{n} d\theta + i d\vec{v} \tag{18}$$

the infinitesimal Lorentz transformation is given as

$$\Lambda = 1 + i\vec{M} d\vec{w} + i\vec{N} d\vec{w}^* \, , \tag{19}$$

where the generators M and N satisfy the Lie algebra

$$M_k M_l - M_l M_k = i \sum_m \epsilon_{klm} M_m \tag{20}$$

$$N_k N_l - N_l N_k = i \sum_m \epsilon_{klm} N_m \tag{21}$$

$$M_k N_l - N_l M_k = 0 \, , \tag{22}$$

that is, two independent rotation Lie algebras. An infinitesimal matrix t from a representation of the Lorentz group is then given as

$$t = \mathbf{1}_M \otimes \mathbf{1}_N + i\vec{M} \otimes \mathbf{1}_N d\vec{w} + i\mathbf{1}_M \otimes \vec{N} d\vec{w}^* \, , \tag{23}$$

where $\mathbf{1}_M$ and $\mathbf{1}_N$ are the unit matrices in the spaces of M and N generators.

Thus an irreducible representation of the Lorentz group is determined by two numbers (m, n), each taking non-negative integer or half-integer values. The dimensions of the M and N-generators are then $(2m + 1)$ and $(2n + 1)$ correspondingly. The dimension of the representation is $(2m+1)(2n+1)$.

Direct product of two irreducible representations There exists a similar theorem for the Lorentz group,

$$(j_1, k_1) \otimes (j_2, k_2) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{k=|k_1-k_2|}^{k_1+k_2} \oplus (j, k) . \tag{24}$$

Rotational properties of an irreducible representation of the Lorentz group If we only consider rotations, $d\vec{v} = 0$, the infinitesimal element of a Lorentz group representation (23) becomes

$$g\big|_{d\vec{v}=0} = 1 + i\left(\vec{M} \otimes \mathbf{1}_N + \mathbf{1}_M \otimes \vec{N}\right) \vec{n} d\theta \, , \tag{25}$$

which can be identified as the Kronecker sum (20) of two generators with rotation Lie algebra. A Kronecker sum of generators corresponds to a direct product of their representations.

Thus, under rotations, an irreducible representations (j_1, j_2) of the Lorentz group reduces to a direct sum of irreducible representations of the rotation group (j) with $j = |j_1 - j_2|, \dots, j_1 + j_2$.

Example: a four-vector $\{E, \mathbf{p}\}$ transforms under $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group and under rotations reduces to a direct sum of a scalar E and a vector $\mathbf{p}$.

Parity transformation and irreducible representations of the Lorentz group Parity transformation is the simultaneous change of spatial coordinates,

$$P \begin{pmatrix} t \\ \vec{x} \end{pmatrix} = \begin{pmatrix} t \\ -\vec{x} \end{pmatrix} . \tag{26}$$

The parity transformation, like rotations, reflects our freedom in choosing frames of reference for a description of physical systems and therefore must be included in the group of coordinate transformations in the principle of covariance.

Under the parity transformation the rotation generators do not change, $\vec{J} \to \vec{J}$, while the velocity-boost-generators change sign, $\vec{K} \to -\vec{K}$. The "optimal" generators $\vec{M} = \frac{1}{2}(\vec{J} - i\vec{K})$ and $\vec{N} = \frac{1}{2}(\vec{J} + i\vec{K})$ transform into each other, $\vec{M} \leftrightarrow \vec{N}$.

Thus an irreducible representation of the Lorentz group (j_1, j_2) transforms under parity transformation into a representation (j_2, j_1),

$$(j_1, j_2) \overset{\mathcal{P}}{\longleftrightarrow} (j_2, j_1) \, . \tag{27}$$

Consequently, if $j_1 \neq j_2$, the representation of the covariance group of coordinate transformations has to be enlarged to the direct sum $(j_1, j_2) \oplus (j_2, j_1)$.