Transformational properties of fields

It was relatively easy to build a covariant theory of a scalar (one-component) field. For multi-component fields (like the electro-magnetic field) we need to learn their transformational properties under coordinate transformations.

Group of coordinate transformations

The coordinate transformations between inertial frames, $x \to x' = ax$, where a is the transformation matrix, form a group, where the group operation is composition; the identity element is identical transformation; and the inverse element is the inverse transformation. In special relativity the group of admissible coordinate transformations is called *Poicaré group* and includes velocity boosts, rotations, translations and reflections. The *Lorentz group* is its subgroup made of rotations and velocity boosts (that is, continuous linear transformations which leave the origin fixed).

Under a coordinate transformation from the Lorentz group[1] $x \to x' = ax$ an n-component physical quantity ψ transforms via some $n \times n$ matrices t_a, $\psi \to \psi' = t_a\psi$. Matrices $\{t_a\}$ form a group *homomorphic* to the group $\{a\}$ of coordinate transformations. Indeed the group operations are apparently preserved,

$$t_{a_1 a_2} = t_{a_1} t_{a_2} \ , \ t_{a^{-1}} = (t_a)^{-1} \ , \ t_{a=\mathbf{1}} = \mathbf{1} \ . \quad (1)$$

The group $\{t_a\}$ is referred to as a *representation* of the group of coordinate transformation.

Lie groups and Lie algebras

The transformation matrices for rotations and velocity boosts are differentiable functions of the transformation parameters (correspondingly, rotation angle and velocity).

A group $\{g\}$ whose elements are differentiable functions of of continuous parameters, $g(\alpha_1, ..., \alpha_n)$, is called a *Lie group*.

An element close to unity (assuming $g(0) = \mathbf{1}$) can be expressed in terms of the *generators* I_k,

$$g(d\alpha) = 1 + i \sum_{k=1}^{n} I_k d\alpha_k \ . \quad (2)$$

[1] translations do not change the fields and therefore can be safely omitted; reflections are discrete transformations and will be considered separately later.

The commutation relation of the generators is called *Lie algebra*,

$$I_j I_m - I_m I_j = i \sum_{k=1}^{n} C^k_{jm} I_k \ , \quad (3)$$

where C^k_{jm} are the so called the *structure constants*. The Lie algebra largely defines the properties of a Lie group.

Lie algebra of rotation group

The transformation of the coordinates under a rotation around z-axis over the angle θ is given as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ . \quad (4)$$

For an infinitesimally small angle $d\theta$ the rotation matrix can be written as

$$1 + i \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\theta \equiv 1 + iI_z d\theta \ , \quad (5)$$

where I_z is the generator of an infinitesimal rotation around z-axis. The corresponding generators I_x and I_y are

$$I_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} , \ I_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} . \quad (6)$$

Direct calculation shows that these generators have the Lie algebra

$$I_k I_l - I_l I_k = i \sum_m \epsilon_{klm} I_m \ , \quad (7)$$

where ϵ_{klm} is the antisymmetric symbol[2] (also called Levi-Civita symbol) of rank 3.

Rotation group is customarily denoted SO(3): special (determinant=1) orthogonal matrix 3×3.

Lie algebra of the Lorentz group

The Lorentz group consists of rotations and velocity boosts. The 4-dimensional rotation generators can be written immediately from (5) and (6) as

$$J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

[2] ϵ_{jkl} is equal +1 (-1) if (j, k, l) is an even (odd) permutation of $(1, 2, 3)$, otherwise it is equal zero.

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} . \quad (8)$$

The Lorentz boost matrix for an infinitesimally small relative velocity dv along one of the axes is

$$\begin{pmatrix} 1 & -dv \\ -dv & 1 \end{pmatrix} = 1 + i \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} dv . \quad (9)$$

Thus the three generators of velocity boosts are

$$K_z = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \; K_y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_x = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \quad (10)$$

Direct calculation gives the Lie algebra of the Lorentz group,

$$J_k J_l - J_l J_k = i \sum_m \epsilon_{klm} J_m \quad (11)$$

$$J_k K_l - K_l J_k = i \sum_m \epsilon_{klm} K_m \quad (12)$$

$$K_k K_l - K_l K_k = -i \sum_m \epsilon_{klm} J_m . \quad (13)$$

The infinitesimal group element corresponding to a rotation around direction $\vec{n}$ with angle $d\theta$, and a velocity boost $d\vec{v}$, is given as[3]

$$g = \mathbf{1} + i\vec{J}\vec{n}d\theta + i\vec{K}d\vec{v} . \quad (14)$$

The Lie algebra of the Lorentz group can be written in a more symmetric way with the (complex) parameterization

$$d\vec{w} = \vec{n}d\theta + id\vec{v} . \quad (15)$$

The infinitesimal Lorentz transformation in is then

$$g = 1 + i\vec{M}d\vec{w} + i\vec{N}d\vec{w}^* , \quad (16)$$

where the (hermitian) generators $\vec{M}$ and $\vec{N}$ are linear combinations of generators $\vec{J}$ and $\vec{K}$

$$\vec{M} = \frac{1}{2}(\vec{J} - i\vec{K}) , \; \vec{N} = \frac{1}{2}(\vec{J} + i\vec{K}) . \quad (17)$$

The Lie algebra for the new generators is

$$M_k M_l - M_l M_k = i \sum_m \epsilon_{klm} M_m , \quad (18)$$

$$N_k N_l - N_l N_k = i \sum_m \epsilon_{klm} N_m , \quad (19)$$

$$M_k N_l - N_l M_k = 0 . \quad (20)$$

Thus the Lorentz Lie algebra is a combination of two independent rotation Lie algebras.

[3]where $\vec{a}\vec{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3$

Irreducible representations of the rotation group

Instead of the generators I_x, I_y, I_z we shall use another parameterization, I_+, I_-, I_z, where

$$I_\pm = \frac{1}{\sqrt{2}}(I_x \pm iI_y), \quad (21)$$

with the commutation relations

$$I_z I_\pm - I_\pm I_z = \pm I_\pm , \; I_+ I_- - I_- I_+ = I_z . \quad (22)$$

Since I_z is hermitian, it has a set of real eigenvalues λ with the corresponding eigenvectors $|\lambda\rangle$,

$$I_z|\lambda\rangle = \lambda|\lambda\rangle . \quad (23)$$

From the commutation relations (22) it follows that the states $I_\pm|\lambda\rangle$ are also eigenvectors of I_z,

$$I_z(I_\pm|\lambda\rangle) = (\lambda \pm 1)(I_\pm|\lambda\rangle). \quad (24)$$

For a finite dimension representation there must exist the largest eigenvalue, say j, such that

$$I_+|j\rangle = 0. \quad (25)$$

Similarly, there should also exist the smallest eigenvalue, such that

$$(I_-)^{(N+1)}|j\rangle = 0, \quad (26)$$

where N is some integer number.

Thus the eigenvalues of I_z is the sequence

$$j, j-1, j-2, ..., j-N. \quad (27)$$

The trace of the generator I_z is equal zero[4],

$$\begin{aligned} \mathrm{trace}(I_z) &= j + (j-1) + ... + (j-N) \\ &= \frac{1}{2}(2j-N)(N+1) = 0 . \end{aligned} \quad (28)$$

Thus $j = N/2$ (either integer or half-integer) and the eigenvalues of I_z are $j, j-1, ..., -j$. The dimension of a representation with a given j is $2j+1$.

The quantities that transform under the irreducible representations (j) of the rotation group are the spherical tensors which are irreducible combinations of normal Euclidean tensors. Half integer j correspond to objects called *spinors*.

[4]taking trace of the commutation relation