Introduction

The subject of quantum field theory

In the context of particle physics a Quantum Field Theory is a theory of elementary (that is, structureless) particles, like electrons and photons.

Elementary particles exhibit wave-particle duality: on the one hand they diffract and interfere as waves in certain fields; on the other hand they appear and disappear as whole entities, called quanta. Hence the name of the theory.

A quantum field theory seeks to explain certain fundamental experimental observations (like the existence of antiparticles; the spin-statistics relation; the CPT symmetry), as well as predict the results of any given experiment (like the cross-section for the Compton scattering, or the value of the anomalous magnetic moment of the electron).

Wave-particle duality

The whole wealth of observations in particle physics supports the concept of wave-particle duality: the elementary particles exhibit both wave and particle properties. On the one hand they are generated and anihilated as whole entities. On the other hand they show diffraction patterns in the double-slit type experiments.

There are two approaches to deal with such objects. One is the path integral formulation, where elementary particles have the property of being able to propogate simultaneously along all possible trajectories with certain apmplitudes. In the other approach, called quantum field theory, elementary particles are field quanta: necessarily chunked ripples in a field.

In the end, the two formulations proved to be equivalent.

We shall pursue the second approach here and build the quantum field theory in the canonical way: as a classical Lagrangian field theory with the subsequent canonical quantization[1].

Principle of covariance

The principle of covariance emphasizes formulation of physical laws using only those physical quantities the measurements of which the observers in different frames of reference could unambiguously correlate. Mathematically speaking, the physical quantities must transform covariantly, that is, under a certain representation of the group of coordinate transformations between admissible frames of reference of the physical theory. This group of coordinate transformations is referred to as the covariance group of the theory.

The principle of covariance does not require invariance of physical laws under the group of admissible transformations although in most cases the equations are actually invariant. Only in the theory of weak interactions the equations are not invariant under reflections (but are, of course, still covariant).

In canonical quantum field theory the admissible frames of reference are the inertial frames of special relativity. The transformations between frames are the Lorentz velocity boosts, rotations, translations, and reflections. Altogether they form the Poincaré group of coordinate transformations. Boosts and rotations together make up the Lorentz group.

The convariant quantities are four-scalars, four-vectors etc. of the Minkowski space of special relativity (and also more complicated objects like bispinors and others which we shall discuss later).

Covariant vectors of special relativity

Four-coordinates

An *event* in an inertial frame can be specified with four coordinates $\{t, \mathbf{r}\}$, where t is the time of the event, and $\mathbf{r} \equiv \{x, y, z\}$ are the three Euclidean spatial coordinates.

The coordinates of the same event in different inertial frames are connected by a linear transformation from the Poincaré group. Rotations, translations and reflections do not couple time and spatial coordinates, but velocity boosts do. Therefore in special relativity time and spatial coordinates are inseparable components of one and the same object. It is only in the non-relativistic limit that time separates from space.

The Lorentz transformation of coordinates under a velocity boost v along the x-axis is given as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} , \quad (1)$$

where primes denote coordinates in the boosted frame, $\gamma \equiv (1 - v^2/c^2)^{-1/2}$, and c is the speed of light in vacuum[2].

The four-coordinates $\{t, \mathbf{r}\}$ are customarily denoted as x^a, where $a = 0 \ldots 3$, such that

$$x^0 = t,\ x^1 = x,\ x^2 = y,\ x^3 = z\ . \quad (2)$$

[1] sometimes unfortunately called "second quantization".

[2] in the following the notation with $\hbar = c = 1$ shall be mostly used.

The Lorentz transformation (1) can then be conveniently written as[3]

$$x'^a = \Lambda^a_b x^b \ . \tag{3}$$

where Λ^a_b is the 4×4 transformation matrix in equation (1).

Invariant

A direct calculation shows that velocity boosts (together with rotations, translations, and reflections) conserve the following form,

$$s^2 = t^2 - \mathbf{r}^2 \ , \tag{4}$$

which is then called *invariant*.

In its infinitesimal incarnation,

$$ds^2 = dt^2 - d\mathbf{r}^2 \ , \tag{5}$$

the form determines the geometry of time-space and is called *metric*. The space with metric (5) is called Minkowski space.

Dual coordinates

The invariant (4) can be conveniently written as

$$t^2 - \mathbf{r}^2 \equiv x_a x^a \ , \tag{6}$$

where x_a are often called *dual coordinates* and are defined as

$$x_a \equiv \{t, -\mathbf{r}\} = g_{ab} x^b \ , \tag{7}$$

where the diagonal tensor g_{ab} with the main diagonal $\{1, -1, -1, -1\}$ is the metric tensor of the Minkowski space of special relativity.

Under Lorentz transformations the dual coordinates apparently transform with the Lorentz matrix where v is substituted by $-v$, which is actually the inverse Lorentz matrix (prove it),

$$x'_a = (\Lambda^{-1})^b_a x_b \ , \tag{8}$$

Four-gradient

The partial derivatives of a scalar with respect to four-coordinates,

$$\partial_a \equiv \frac{\partial}{\partial x^a} = \left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} \equiv \left\{\frac{\partial}{\partial t}, \nabla\right\} \ , \tag{9}$$

apparently transform like dual coordinates, that is, via the inverse Lorentz matrix,

$$\frac{\partial}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a}\frac{\partial}{\partial x^b} = (\Lambda^{-1})^b_a \frac{\partial}{\partial x^b} \ . \tag{10}$$

[3] note the "implicit summation" notation, $\Lambda^a_b x^b \equiv \sum_{b=0}^{3} \Lambda^a_b x^b$.

Covariant vectors and tensors

A *contra-variant four-vector* is a set of four objects, $A^a = \{A^0, \mathbf{A}\}$, which transform from one inertial frame to another in the same way as coordinates in (3),

$$A'^a = \Lambda^a_b A^b \ . \tag{11}$$

A *co-variant four-vector* is a set of four objects, A_a, which transform from one inertial frame to another in the same way as partial derivatives in (10),

$$A'_a = (\Lambda^{-1})^b_a A_b \ . \tag{12}$$

A *covariant tensor* F^{ab} is a set of 4×4 objects which transform between inertial frames as a product of two 4-vectors,

$$F'^{ab} = \Lambda^a_c \Lambda^b_d F^{cd} \ . \tag{13}$$

There exist other covariant objects in special relativity, like bispinors, which cannot be built out of 4-vectors. They will be discussed later.

Exercises

1. Derive the Lorentz transformation matrix,
$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{bmatrix} 1 & \frac{-v}{c^2} \\ -v & 1 \end{bmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \ , \tag{14}$$
one way or another, for example using group postulates and the existence of the highest relative velocity between inertial frames (see wikipedia).

2. Show that the interval
$$ds^2 = c^2 dt^2 - d\mathbf{r}^2$$
is invariant under Lorentz transformations, rotations, translations and reflections.

3. Show that a moving clock runs slower, than stationary. Hint: consider the transformation of
$$\begin{pmatrix} dt \\ d = 0 \end{pmatrix} .$$

4. Show that a moving rod is shorter, than stationary. Hint: consider a transformation into
$$\begin{pmatrix} dt' = 0 \\ dx' \end{pmatrix} .$$

5. Show that in the limit $v \ll c$ the Lorentz transformation reduces to Galilean transformation.

6. Which of the following objects are covariant?
   - Kronecker delta $\delta^a_b \equiv \{a == b ? 1 : 0\}$;
   - Lorentz transformation matrix Λ^a_b in (1);
   - Metric tensor g_{ab}