Exercises

1. Show that if the operators

   $$U = \exp\left(-i\sum_k J_k\alpha_k\right)$$

   are unitary: $U^\dagger U = 1$, and the parameters α_k are real, then the generators J_k are hermitian: $J_k^\dagger = J_k$. Show that if $\mathsf{det}(U) = 1$ then $\mathsf{trace}(J_k) = 0$.

   Hints: consider infinitesimal elements; show that $\mathsf{det}(e^A) = e^{\mathsf{trace}(A)}$ by diagonalizing matrix A.

2. Using the Lie algebra of the rotation group[1]

   $$[I_j, I_k] = i\sum_l \epsilon_{jkl} I_l$$

   find the 2x2 representation of the rotation generators $\vec{I}$ (assuming I_3 is diagonal and I_1 is real).

   Hints: recall that $\mathsf{trace}(I_k) = 0$ and $I_k^+ = I_k$; I's must be half the Pauli's σ-matrices.

[1] ϵ_{jkl} is the Levi-Civita (absolutely antisymmetric) tensor