Interacting quantum fields

Interaction Lagrangian

If two fields, say ψ and ϕ, interact, the total Lagrangian $\mathcal{L}$ for the combined system is equal the sum of the free Lagrangians, $\mathcal{L}_\psi$ and $\mathcal{L}_\phi$, plus some interaction Lagrangian $\mathcal{L}_v$,

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_\phi + \mathcal{L}_v \; . \tag{1}$$

The interaction Lagrangian $\mathcal{L}_v$ contains both fields ψ and ϕ. For example, the interaction between a bispinor field ψ and a real scalar field ϕ can be

$$L_v = -g\bar{\psi}\psi\phi \; , \tag{2}$$

where g is the coupling constant.

The dimension of the coupling constant is of importance for the perturbation theory. The theory works best (can be renormalized) only if the coupling constant is dimensionless. In the units $\hbar = c = 1$ all dimensions are expressed in terms of mass m,

$$S \sim Et \sim 1 \; , \tag{3}$$
$$t \sim x \sim m^{-1} \; , \tag{4}$$
$$\mathcal{L} \sim m^4 \; , \tag{5}$$
$$\phi \sim m \; , \tag{6}$$
$$\psi \sim m^{3/2} \; , \tag{7}$$

where S is action, ϕ is spin-0 field, and ψ is spin-1/2 field. The coupling constant in the interaction Lagrangian (2) is dimensionless.

The Hamiltonian for the system of interacting fields will be a sum of Hamiltonians of the free fields, H_0, and some interacting potential V,

$$H = H_0 + V \; . \tag{8}$$

In the following we shall use interaction Lagrangians containing fields but not the field derivatives. For such Lagrangians

$$V = -\int d^3x \mathcal{L}_v \; . \tag{9}$$

Time development in quantum mechanics

In quantum mechanics we postulate that the time evolution of a matrix operator O (let's assume – not explicitely depending on time, for simplicity) follows the Hamilton equation

$$\frac{\partial O}{\partial t} = \frac{1}{i}[O, H] \; , \tag{10}$$

where $\frac{1}{i}[O,H] \equiv \frac{1}{i}(OH - HO)$ is the (generalized) Poisson bracket for matrices and H is the Hamiltonian of the system.

Considering the elements of O as matrix elements of some operator $\hat{O}$, $\langle \Phi_b|\hat{O}|\Phi_a\rangle$, the time dependence in (10) can be attributed to the operator $\hat{O}$, the state vectors $\langle\Phi_b|$ and $|\Phi_a\rangle$, or to both. Hence we can have the following "pictures" for the time-developement of a quantum system with the Hamiltonian $H = H_0 + V$,

1. Heisenberg picture
$$i\frac{\partial \Phi}{\partial t} = 0 \; , \; i\frac{\partial O}{\partial t} = [O, H] \tag{11}$$
2. Schrödinger picture
$$i\frac{\partial \Phi}{\partial t} = H\Phi \; , \; i\frac{\partial O}{\partial t} = 0 \tag{12}$$
3. Interaction picture
$$i\frac{\partial \Phi}{\partial t} = V\Phi \; , \; i\frac{\partial O}{\partial t} = [O, H_0] \; . \tag{13}$$

We shall use the interaction picture where the quantum state of a system is time-independent for free fields. A weak interaction would lead to slow transitions between the states of free fields (normal modes).

S-matrix

Let us try to formally integrate the time evolution equation $i\frac{\partial\Phi}{\partial t} = V\Phi$ using small time step Δt:

$$\Phi(t_0 + \Delta t) = (1 - iV(t_0)\Delta t)\Phi \; . \tag{14}$$

After N time steps

$$\Phi(t_0 + N\Delta t) = \big(1 - iV(t_{N-1})\Delta t\big) \cdot$$
$$\cdot\big(1 - iV(t_{N-2})\Delta t\big) \cdot \ldots \cdot \big(1 - iV(t_0)\Delta t\big)\Phi(t_0) \tag{15}$$

Note that the later time V's come at the left. We cannot exchange the order of V's since they contain generation/annihilation operators which do not commute. The later sum can be rewritten as

$$\Phi(t_0 + Ndt) = \Big(1 + \frac{(-i)}{1!}\sum_i V(t_i)\Delta t0 +$$
$$+\frac{(-i)^2}{2!}\sum_{ij} T\big(V(t_i)V(t_j)\big)\Delta t^2 + \ldots\Big)\Phi(t_0), \tag{16}$$

where the time ordering operator T arranges the V's according to their times, the older V's to the left,

$$T\big(V(t_1)V(t_2)\big) = \begin{cases} V(t_1)V(t_2) & \text{, if } t_1 > t_2 \\ V(t_2)V(t_1) & \text{, if } t_1 < t_2 \end{cases} \tag{17}$$

In the limit $t_0 \to -\infty$, $N \to \infty$ the time evolution will be

$$\Phi(+\infty) = \Big(\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \times$$
$$\times \int T\big(V(t_1)\dots V(t_n)\big)dt_1\dots dt_n\Big)\Phi(-\infty) \quad (18)$$

S-matrix is the operator that performs time evolution of the state-vector from $t = -\infty$ to $t = +\infty$,

$$\Phi(+\infty) = S\Phi(-\infty) \ . \quad (19)$$

From (18)

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int T\big(V(t_1)\dots V(t_n)\big)dt_1\dots dt_n \\ &\equiv T\exp\left(-i\int_{-\infty}^{+\infty} V(t)dt\right) \quad (20) \\ &= T\exp\left(i\int d^4x\mathcal{L}_v\right) \quad (21) \\ & \quad (22) \end{aligned}$$

The probability $P_{f\leftarrow i}$ of a transition from the initial state $|i\rangle$ to the final state $|f\rangle$ is given by

$$P_{f\leftarrow i} = |\langle f|S|i\rangle|^2 \equiv |S_{fi}|^2. \quad (23)$$

Exercises

1. Consider two "classical" interacting fields, a complex spin-1/2 field ψ and a real scalar field ϕ with interacting Lagrangian $\mathcal{L}_v = -g\bar{\psi}\psi\phi$. Find the Euler-Lagrange equations and the Hamiltonian density T_0^0 of the system.

2. Show that (for a real scalar field $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$) commutation relations $[a_k, a_k^\dagger] = 1$ for the generation/annihilation operators lead to the commutation relation $[\chi, \pi] = i$ for a canonical generalized coordinate χ and the corresponding momentum π. Hints:

   (a) For a field the generalized coordinates are the values of the field itself, $\chi = \phi$. According to the general rule the generalized momentum is $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\chi}}$. Find the expression for π (answer: $\pi = \partial_0\phi$).

   (b) Consider the amplitude of a single plane wave as the generalized (now operator) coordinate

   $$\chi = \phi_k = \frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}e^{-ikx} + a_{\vec{k}}^\dagger e^{+ikx}\right)$$

   and calculate the corresponding operator π.

   (c) Calculate the commutator $[\chi, \pi]$.

3. Time evolution in Heisenberg-Born-Jordan matrix mechanics: show that the canonical (Hamilton) equations of motion

$$\frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q} \ , \ \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p} \ , \ \frac{\partial f(q,p)}{\partial t} = \{f, H\} \ ,$$

(where

$$\{f, H\} \equiv \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}$$

is the classical Poison bracket) with the commutation relation $[q, p] = i$ lead to operator equations

$$\frac{\partial p}{\partial t} = \frac{1}{i}[p, H] \ , \ \frac{\partial q}{\partial t} = \frac{1}{i}[q, H] \ , \ \frac{\partial f}{\partial t} = \frac{1}{i}[f, H] \ .$$

Hint: show (by induction) that

$$\frac{1}{i}[q, f] = \frac{\partial f}{\partial p}$$

$$\frac{1}{i}[p, f] = -\frac{\partial f}{\partial q}$$

for any function which can be represented as a series of powers of q and p.