A group element in terms of generators: finite rotation matrix

Having the generators one can recover not only infinitesimal, $g(d\alpha) = 1 + iId\alpha$, but (in principle) also finite elements of the group (with certain caveats). It is especially easy with additive parameters, where $g(\alpha + d\alpha) = g(\alpha)g(d\alpha)$.

For example, for a rotation around a given axis $\mathbf{n}$ the rotation angle θ is an additive parameter of the rotation matrix $R(\mathbf{n}, \theta)$,

$$\begin{aligned} R(\mathbf{n}, \theta + d\theta) &= R(\mathbf{n}, \theta)R(\mathbf{n}, d\theta) \\ &= R(\mathbf{n}, \theta)(1 + i\mathbf{In}d\theta) , \end{aligned} \quad (1)$$

and thus the rotation matrix satisfies the differential equation

$$\frac{dR}{d\theta} = iR\,\mathbf{In} , \quad (2)$$

with the boundary condition $R(\mathbf{n}, \theta = 0) = 1$. The solution is given by the Taylor expansion,

$$R(\mathbf{n}, \theta) = e^{i\mathbf{In}\theta} \equiv 1 + i\mathbf{In}\theta + \frac{1}{2!}(i\mathbf{In}\theta)^2 + \ldots . \quad (3)$$

In practice for a finite-dimension-representation there is only a finite number of terms in the sum.

Direct product of two representations of a group

The generators $I_k^{(1\otimes 2)}$ of a direct product $g^{(1)} \otimes g^{(2)}$ of two representations $g^{(1)}$ and $g^{(2)}$ of a group is a direct sum of the corresponding generators $I_k^{(1)}$ and $I_k^{(2)}$,

$$\begin{aligned} g^{(1)} \otimes g^{(2)} &= \left(1^{(1)} + iI_k^{(1)}\alpha_k\right) \otimes \left(1^{(2)} + iI_k^{(2)}\alpha_k\right) \\ &= 1^{(1)} \otimes 1^{(2)} + i\left(1^{(2)} \otimes I_k^{(1)} + 1^{(1)} \otimes I_k^{(2)}\right)\alpha_k , \\ &\Rightarrow I_k^{(1\otimes 2)} = I_k^{(1)} \oplus I_k^{(2)} . \end{aligned} \quad (4)$$

Direct product of two irreducible representations of the rotation group (Clebsch-Gordan theorem)

The direct product $(j_1) \otimes (j_2)$ of two irreducible representations of the rotation group is a reducible representation. It can be reduced into a direct sum of irreducible representations,

$$(j_1) \otimes (j_2) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \oplus(j) . \quad (5)$$

Example: a direct product $\mathbf{a} \otimes \mathbf{b}$ of two vectors, $(1) \otimes (1)$, reduces to a direct sum of a scalar $(j = 0)$, an antisymmetric tensor $(j = 1)$, and a symmetric tensor with zero trace $(j = 2)$,

$$\mathbf{a} \otimes \mathbf{b} = (\mathbf{ab}) \oplus (\mathbf{a} \times \mathbf{b}) \oplus \left(a_i b_j + a_j b_i - \frac{2}{3}(\mathbf{ab})\delta_{ij}\right) . \quad (6)$$

Irreducible representations of the Lorentz group

With the complex parametrization

$$d\mathbf{w} = \mathbf{n}d\theta + id\mathbf{v} \quad (7)$$

the infinitesimal Lorentz transformation is given as

$$g = 1 + i\mathbf{M}d\mathbf{w} + i\mathbf{N}d\mathbf{w}^* , \quad (8)$$

where the generators M and N satisfy the Lie algebra

$$\begin{aligned} M_k M_l - M_l M_k &= i\epsilon_{klm} M_m & (9) \\ N_k N_l - N_l N_k &= i\epsilon_{klm} N_m & (10) \\ M_k N_l - N_l M_k &= 0 , & (11) \end{aligned}$$

which is apparently two independent rotation Lie algebras. Thus an irreducible representation of the Lorentz group is determined by two numbers (j_1, j_2), each taking (non-negative) integer or half-integer values.

Direct product of two irreducible representations of the Lorentz group

There exists a similar theorem for the Lorentz group,

$$(j_1, j_2) \otimes (j_1' j_2') = \sum_{k_1=|j_1-j_1'|}^{j_1+j_1'} \sum_{k_2=|j_2-j_2'|}^{j_2+j_2'} \oplus(k_1, k_2) . \quad (12)$$

Rotational properties of an irreducible representation of the Lorentz group

If we only consider rotations, $d\mathbf{v} = 0$, the infinitesimal element of a Lorentz group representation (8) becomes

$$g(d\mathbf{v} = 0) = 1 + i\left(\mathbf{M} + \mathbf{N}\right)\mathbf{n}d\theta , \quad (13)$$

which can be identified as a direct sum of two generators with rotation Lie algebra (9), corresponding to a direct product of two representations of the rotations group.

Thus, under rotations only, an irreducible representations (j_1, j_2) of the Lorentz group reduces to a direct sum of irreducible representations of the rotation group (j) with $j = |j_1 - j_2|, \ldots, j_1 + j_2$.

Example: a four-vector $\{E, \mathbf{p}\}$ transforms under $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group and under rotations reduces to a direct sum of a scalar E and a vector $\mathbf{p}$.

Parity transformation

Parity transformation is the simultaneous change of spatial coordinates,

$$P\begin{pmatrix} t \\ \vec{x} \end{pmatrix} = \begin{pmatrix} t \\ -\vec{x} \end{pmatrix} . \tag{14}$$

The parity transformation, like rotations, reflects our freedom in choosing frames of reference for a description of physical systems and therefore must be included in the group of coordinate transformations in the principle of covariance.

Under the parity transformation the rotation generators do not change, $\vec{J} \to \vec{J}$, while the velocity-boost-generators change sign, $\vec{K} \to -\vec{K}$. The "optimal" generators $\vec{M} = \frac{1}{2}(\vec{J} - i\vec{K})$ and $\vec{N} = \frac{1}{2}(\vec{J} + i\vec{K})$ transform into each other, $\vec{M} \leftrightarrow \vec{N}$.

Thus an irreducible representation of the Lorentz group (j_1, j_2) transforms under parity transformation into a representation (j_2, j_1). Consequently, if $j_1 \neq j_2$, the representation of the covariance group of coordinate transformations has to be enlarged to the direct sum $(j_1, j_2) \oplus (j_2, j_1)$.

Exercises

1. Find the finite rotation matrix for a 2×2 representation of the rotation group[1].

2. Using the additive parameter $\varphi = \frac{1}{2} \ln \frac{1+v}{1-v}$ find the velocity-boost-matrices for the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group.

3. Prove that generators of a Lie group have a Lie algebra

   $$[I_l, I_m] \equiv I_l I_m - I_m I_l = i C^k_{lm} I_k ,$$

   where C^k_{lm} are some (structure) constants[2].

[1] Hint: $\vec{I} = \frac{1}{2}\vec{\sigma}$, and $(\vec{\sigma}\vec{n})^2 = 1$, where $\vec{\sigma}$ are Pauli matrices.

[2] Hint: consider a group element

$$g(\beta') = g(\alpha) g(\beta) g^{-1}(\alpha) ,$$

in the limit $\beta \to 0$ and $\alpha \to 0$, namely:

$$\lim_{\beta \to 0} \beta'_k = f_{kl}(\alpha) \beta_l ,$$

where $f_{kl}(\alpha)$ are some functions with

$$\lim_{\alpha \to 0} f_{kl}(\alpha) = \delta_{kl} + C^k_{lm} \alpha_m .$$