## Transformational properties of fields

It was relatively easy to build a covariant theory for a scalar (one-component) field. For multi-component fields (like the electro-magnetic field) we need to learn their transformational properties under coordinate transformations.

### The group of coordinate transformations

The coordinate transformations $x \to x' = ax$ between inertial frames form a group: the group operation is composition, the identity element is identical transformation, and the inverse element is the inverse transformation.

An $n$-component physical quantity $\psi$ transforms with some $n \times n$ matrices $t_a$, $\psi \to \psi' = t_a\psi$, which form a group homomorphic to the group of coordinate transformations, since the group operations are apparently preserved,

$$t_{a_1 a_2} = t_{a_1} t_{a_2}\,,\; t_{a^{-1}} = (t_a)^{-1},\; t_{a=\mathbf{1}} = \mathbf{1}\;. \qquad (1)$$

The group $t_a$ is referred to as a *representation* of the group of coordinate transformation.

### Lie groups and Lie algebras

For rotations and velocity boosts the transformation matrices are differentiable functions of the transformation parameters.

A group whose elements $g$ are differentiable functions of a set of continuous parameters, $g(\alpha_1, ..., \alpha_n)$, is called a *Lie group*.

An element close to 1 (assuming $g(0) = 1$) can be expressed in terms of the *generators* $I_k$,

$$g(d\alpha) = 1 + i \sum_{k=1}^{n} I_k d\alpha_k \qquad (2)$$

The commutation relations of the generators are called *Lie algebra*,

$$I_j I_m - I_m I_j = i \sum_{k=1}^{n} C^k_{jm} I_k, \qquad (3)$$

where $C^k_{jm}$ is the so called the *structure constant*. The Lie algebra largely defines the properties of a Lie group.

### Lie algebra of the rotation group

The transformation of the coordinates under a rotation around $z$-axis over the angle $\theta$ is given as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \qquad (4)$$

For an infinitesimally small angle $d\theta$ the rotation matrix can be written as

$$1 + i \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\theta \equiv 1 + iI_z d\theta\;, \qquad (5)$$

where $I_z$ is the generator of an infinitesimal rotation around $z$-axis. The corresponding generators $I_x$ and $I_y$ are

$$I_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},\; I_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} . \qquad (6)$$

Direct calculation shows that these generators have the Lie algebra

$$I_k I_l - I_l I_k = i \sum_m \epsilon_{klm} I_m\;, \qquad (7)$$

where $\epsilon_{klm}$ is an antisymmetric tensor of rank 3.

Rotation group is denoted SO(3), which stands for special (determinant 1) orthogonal matrix 3×3.

### The Lie algebra of the Lorentz group

The Lorentz group consists of rotations and velocity boosts. The 4-dimensional rotation generators can be written immediately from (5) and (6) as

$$J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} . \qquad (8)$$

The Lorentz boost matrix for an infinitesimally small relative velocity $dv$ along one of the axes is

$$\begin{pmatrix} 1 & -dv \\ -dv & 1 \end{pmatrix} = 1 + i \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} dv\;. \qquad (9)$$

Thus the three generators of velocity boosts are

$$K_z = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} K_y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_x = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (10)$$

Having the generators, one can calculate the Lie algebra of the Lorentz group

$$J_k J_l - J_l J_k = i \sum_m \epsilon_{klm} J_m \quad (11)$$

$$J_k K_l - K_l J_k = i \sum_m \epsilon_{klm} K_m \quad (12)$$

$$K_k K_l - K_l K_k = -i \sum_m \epsilon_{klm} J_m \ . \quad (13)$$

The infinitesimal group element corresponding to a rotation around the direction $\vec{n}$ over an angle $d\theta$ and a velocity boost $d\vec{v}$ is given as[1]

$$g = 1 + i\vec{J}\vec{n}d\theta + i\vec{K}d\vec{v}. \quad (14)$$

The Lie algebra of the Lorentz group can be written in a more symmetric way with a (complex) parametrization

$$d\vec{w} = \vec{n}d\theta + id\vec{v}. \quad (15)$$

The infinitesimal Lorentz transformation is then

$$g = 1 + i\vec{M}d\vec{w} + i\vec{N}d\vec{w}^* \ , \quad (16)$$

where the (hermitian) generators $\vec{M}$ and $\vec{N}$ are linear combinations of generators $\vec{J}$ and $\vec{K}$

$$\vec{M} = \frac{1}{2}(\vec{J} - i\vec{K}) \ , \ \vec{N} = \frac{1}{2}(\vec{J} + i\vec{K}) \ . \quad (17)$$

The Lie algebra for the new generators is

$$M_k M_l - M_l M_k = i \sum_m \epsilon_{klm} M_m \quad (18)$$

$$N_k N_l - N_l N_k = i \sum_m \epsilon_{klm} N_m \quad (19)$$

$$M_k N_l - N_l M_k = 0. \quad (20)$$

Thus the Lorentz Lie algebra is a combination of two independent rotation Lie algebras.

### Irreducible representations of the rotation group

Canonically, instead of the generators $I_x$, $I_y$, $I_z$ we shall use another parametrization, $I_+$, $I_-$, $I_z$, where

$$I_\pm = \frac{1}{\sqrt{2}}(I_x \pm I_y), \quad (21)$$

with the commutation relation

$$I_z I_\pm - I_\pm I_z = \pm I_\pm \ , \ I_+ I_- - I_- I_+ = I_z. \quad (22)$$

[1] where $\vec{a}\vec{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3$

Let us look for the representations where the $I_z$ matrix is diagonal. Let it have eigenvectors $|\lambda\rangle$ with eigenvalues $\lambda$,

$$I_z|\lambda\rangle = \lambda|\lambda\rangle \ . \quad (23)$$

From the commutation relations (22) follows, that the states $I_\pm|\lambda\rangle$ are also eigenvectors of $I_z$,

$$I_z(I_\pm|\lambda\rangle) = (\lambda \pm 1)(I_\pm|\lambda\rangle). \quad (24)$$

For a finite dimension representation there must exist the largest eigenvalue, say $j$, such that

$$I_+|j\rangle = 0. \quad (25)$$

Similarly, there should also exist the smallest eigenvalue, such that

$$(I_-)^{(N+1)}|j\rangle = 0, \quad (26)$$

where $N$ is some integer number.

Thus the eigenvalues of $I_z$ is the sequence

$$j, j-1, j-2, ..., j-N. \quad (27)$$

The trace of the generator $I_z$ is equal zero[2],

$$\begin{aligned} \mathrm{trace}(I_z) &= j + (j-1) + ... + (j-N) \\ &= \frac{1}{2}(2j-N)(N+1) = 0 \ . \end{aligned} \quad (28)$$

Thus $j = N/2$ and the eigenvalues of $I_z$ are $j, j-1, ..., -j$. The dimension of a representation with a given $j$ is $2j+1$.

### Exercises

1. A group parameter $\varphi$ is called additive if $g(\varphi_1)g(\varphi_2) = g(\varphi_1 + \varphi_2)$. Find the additive parameter of the Lorentz boosts along the $z$-axis. Find the corresponding generator.
2. Show that if $U = \exp(-i\sum_k J_k \alpha_k)$ is unitary ($U^\dagger U = 1$) and the parameters $\alpha_k$ are real, then the operators $J_k$ are hermitian ($J_k^\dagger = J_k$). Show that if $\mathsf{det}(U) = 1$ then $\mathsf{trace}(J_k) = 0$.
3. From the commutation relations
$$[I_j, I_k] = i \sum_l \epsilon_{jkl} I_l$$
find the 2x2 representation of the generators $\vec{I}$ (assuming $I_3$ is diagonal and $I_1$ is real).

[2] taking trace of the commutation relation