Classical Lagrangian field theory

When building a theory for a physical system it is often of advantage to use the *variational principle*[1] instead of directly postulating the equations of motions.

Action and Lagrangian

The action $S(\phi)$ of a field ϕ is a functional (it takes a function as an argument and returns a number as a result) of a field $\phi(x^\mu)$ in the form

$$S(\phi) = \int_{\text{some volume}} d^4x \mathcal{L}(\phi, \partial_\mu \phi), \tag{1}$$

where the kernel $\mathcal{L}(\phi, \partial_\mu \phi)$ is called Lagrangian density or simply *Lagrangian.*

Variational Principle

The variational principle is used (among many other useful things) to find the trajectory (a full description of the motion) of a physical system:

> *Of all possible trajectories of a physical system with the given boundary condition the theoretical solution is the one which provides the least action.*

Alternatively,

> *The variation of the action vanishes on the theoretical solution for the trajectory of the system.*

Thus if $\phi(x^\mu)$ is a solution, and $\delta\phi(x^\mu)$ is a small arbitrary variation (which does not change the boundary condition), the (linear part of the) variation of the action vanishes,

$$\delta S(\phi) \equiv S(\phi + \delta\phi) - S(\phi) = 0. \tag{2}$$

Euler-Lagrange equation

The trajectory of minimal action actually satisfies an equation of motion, called Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \tag{3}$$

which can be derived from the variational principle.

[1] also called "Stationary Action Principle" or "Least Action Principle".

Translation invariance and energy-momentum tensor

Since, as the experiment shows, the space-time is uniform or, in other words, *translation invariant*, the Lagrangian can not depend explicitly on coordinates x^ν. Thus, under a translation $x^\nu \to x^\nu + \delta x^\nu$, the variation of the Lagrangian is given by

$$\delta L = \frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial(\partial_\mu \phi)}\partial_\mu \delta\phi \,. \tag{4}$$

Using the equation of motion (3) we can rewrite this as

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)}\delta\phi \right) - \delta L = 0 \tag{5}$$

Dividing the equation by δx^ν gives

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)}\partial_\nu\phi \right) - \partial_\nu L = 0 \;, \tag{6}$$

which can be rewritten in a form of a 4-divergence

$$\partial_\mu T^\mu_\nu = 0 \;, \tag{7}$$

where T^μ_ν is called the *canonical energy-momentum tensor*,

$$T^\mu_\nu = \frac{\partial L}{\partial(\partial_\mu \phi)}\partial_\nu\phi - L\delta^\mu_\nu \;. \tag{8}$$

Conservation of energy and momentum

The vanishing 4-divergence can be rewritten in a 3-notation as

$$\frac{\partial}{\partial t}T^0_\nu = -\frac{\partial}{\partial r^i}T^i_\nu \;, \tag{9}$$

where i=1,2,3. Suppose now that we have an isolated system in a (large) 3-volume V where the field is zero at the surface of the volume and beyond. Integrating (9) over the 3-volume V gives the four conservation laws for the energy and momentum:

$$\frac{\partial}{\partial t}P_\nu = 0 \;, \tag{10}$$

where P_ν is the energy-momentum 4-vector of the field

$$P_\nu = \int d^3x T^0_\nu \tag{11}$$

Thus the time components of the energy-momentum tensor can be interpreted as energy-momentum densities of our field,

$$T^0_0 = \frac{\partial L}{\partial(\partial_t \phi)}\partial_t\phi - L \;, T^0_i = \frac{\partial L}{\partial(\partial_t \phi)}\partial_i\phi \tag{12}$$

The Hamiltonian (that is the energy as function of fields) is the time-component of the energy-momentum 4-vector.

$$H = P_0 = \int d^3x T^0_0 \; . \tag{13}$$

Note that energy and momentum is conserved only on the solutions of the Euler-Lagrange equation.

Gauge invariance and conserved current

If the field ϕ is complex there is another transformation which leaves our Lagrangian invariant – the global change of the phase of the field

$$\phi \to \phi e^{i\epsilon} \approx \phi + i\epsilon\phi, \tag{14}$$

where ϵ is an arbitrary phase shift (small for simplicity). This transformation is often called the global *gauge transformation.*

The variation of the Lagrangian under the phase shift is zero, since it is real (perhaps we need to assume additionally that the Lagrangian contains only integer powers of fields).

$$\delta L = \frac{\partial L}{\partial\phi}\delta\phi + \frac{\partial L}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi$$
$$+\frac{\partial L}{\partial\phi^*}\delta\phi^* + \frac{\partial L}{\partial(\partial_\mu\phi^*)}\partial_\mu\delta\phi^* = 0 \tag{15}$$

Using the Euler-Lagrange equation of motion (3) and the fact that for the gauge transformation $\delta\phi = i\epsilon\phi$ and $\delta\phi^* = -i\epsilon\phi^*$ this can be rewritten as a continuity equation (vanishing 4-divergence),

$$\partial_\mu j^\mu = 0 \; , \tag{16}$$

where

$$j^\mu = \frac{1}{i}\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\phi - \frac{\partial L}{\partial(\partial_\mu\phi^*)}\phi^*\right) \; . \tag{17}$$

is called the *conserved current.*

Charge conservation

The continuity equation (16) can be interpreted in 3-notation, $j^\mu = \{\rho, \mathbf{j}\}$, as the charge conservation law. In the differential form

$$\frac{d\rho}{dt} = -\text{div}\mathbf{j} \; , \tag{18}$$

and in the integral form

$$\frac{dQ}{dt} = -\oint_{boundary(volume)} \mathbf{j}\cdot d\mathbf{s} \; , \tag{19}$$

where Q is the charge inside the volume

$$Q = \int_{volume} dV j^0 \tag{20}$$

If we have an isolated system (i.e. no flux through the boundary) its charge conserves.

Note that for a real field the current is apparently zero – real fields are necessarily neutral.

Exercises

1. Derive the Euler-Lagrange equation,
$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \; ,$$
from the variational principle $\delta S = 0$.
2. Derive the Klein-Gordon equation
$$\partial_\mu\partial^\mu\phi + m^2\phi = 0 \; ,$$
from the Lagrangian
$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \; .$$

Hints

$$\delta S(\phi) = \int_{volume} d^4x \delta L(\phi, \partial_\mu\phi)$$
$$= \int_{volume} d^4x \left(\frac{\partial L}{\partial\phi}\delta\phi + \frac{\partial L}{\partial(\partial_\mu\phi)}\delta\partial_\mu\phi\right)$$
$$= \int_{volume} d^4x \left(\frac{\partial L}{\partial\phi}\delta\phi + \partial_\mu(\frac{\partial L}{\partial(\partial_\mu\phi)}\delta\phi) - \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi)}\delta\phi\right)$$
$$= \int_{volume} d^4x\delta\phi\left(\frac{\partial L}{\partial\phi} - \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi)}\right)$$
$$+\int_{boundary(volume)} d\sigma_\mu\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\delta\phi\right)$$
$$= \int_{volume} d^4x\delta\phi\left(\frac{\partial L}{\partial\phi} - \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi)}\right) = 0$$
$$\Rightarrow \frac{\partial L}{\partial\phi} - \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi)} = 0$$