Gauge theories

Gauge theory is a peculiar quantum field theory where the Lagrangian is invariant under the so called *local gauge transformations*. The transformations form a Lie group which is referred to as the *symmetry group* or *gauge group* of the theory. For each group parameter there is a special vector field, called *gauge field*, which helps to make the Lagrangian group invariant. The quanta of the gauge field are called *gauge bosons*. If the symmetry group is commutative, the gauge theory is called *abelian*, otherwise it is called (surprise, surprise) *non-abelian* or simply *Yang-Mills* theory.

The good old QED happened to be a gauge theory with the group $U(1)$. Then it turned out that only gauge theories can be renormalized and therefore the Standard Model had to be a gauge theory, though in order to incorporate more gauge bosons the gauge group had to be a bit higher than $U(1)$. It turned out $SU(2)$ and $SU(3)$ fit perfectly for the weak and strong interactions correspondingly.

Example: quantum electrodynamics

Quantum electrodynamics (QED) is a theory of the electron/positron (bispinor) field ψ coupled to the electromagnetic (vector) field A_μ with the interaction Lagrangian $-j_\mu A^\mu$, where $j_\mu = g\bar\psi\gamma_\mu\psi$ is the conserved current (g is the charge of electron).

The QED Lagrangian,

$$\mathcal{L} = \bar\psi\gamma^\mu i(\partial_\mu + igA_\mu)\psi - m\bar\psi\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (1)$$

is invariant under the local gauge transformation,

$$\begin{cases} \psi & \rightarrow \quad \psi' = e^{ig\alpha(x)}\psi \\ A_\mu & \rightarrow \quad A'_\mu = A_\mu - \partial_\mu\alpha(x) \end{cases} . \quad (2)$$

where $\alpha(x)$ is an arbitrary scalar function (and also the group parameter). The transformation matrices $e^{ig\alpha}$ form a Lie group $U(1)$ where the charge is the group generator. QED is thus a gauge theory with the symmetry group $U(1)$. The group is commutative, therefore QED is an abelian theory.

The first term of the Lagrangian (1) is conveniently written as

$$\bar\psi\gamma^\mu iD_\mu\psi \quad (3)$$

where the *group covariant derivative*

$$D_\mu = \partial_\mu + igA_\mu \quad (4)$$

has the property that under the gauge transformation

$$D_\mu\psi \rightarrow e^{ig\alpha(x)}D_\mu\psi. \quad (5)$$

Note that

$$F_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu] \; . \quad (6)$$

Non-abelian (Yang-Mills) gauge theories

QED is a very good re-normalizable theory, but it only has one gauge boson, the photon, while you need a lot more for the standard model...

Yang and Mills suggested to consider more general groups of matrices

$$U(x) = e^{iI_a\alpha_a(x)} \quad (7)$$

with some generators I_a, $a = 1\ldots n$, and Lie-algebras

$$[I_a, I_b] = C^c_{ab}I_c. \quad (8)$$

If the structure constants C^c_{ab} are not equal zero, the theory is called non-Abelian.

To make the theory gauge invariant a separate vector field A^μ_a is needed for each group parameter α_a and the gauge field becomes $A^\mu = I^a A^\mu_a$ (there is no difference whether the group index a is up or down). The generalized gauge transformation is now defined as

$$\begin{cases} \psi & \rightarrow \quad \psi' = U\psi \; , \\ A_\mu & \rightarrow \quad A'_\mu = UA_\mu U^{-1} - \frac{1}{ig}\partial_\mu U \cdot U^{-1} \; . \end{cases} \quad (9)$$

The transformation matrices U have certain dimension and thus the field ψ, on which the matrix operates, becomes a column-vector in this new dimension[1].

In a Yang-Mills theory the fermionic field ψ is a non-trivial object: it is an (generation/annihilation) operator in the space of quantum states of the field; it is a Lorentz group bispinor; and it is also an object rotated by a (usually fundamental) representation of the gauge group.

The Yang-Mills field tensor is

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{ig}[D_\mu, D_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \\ &= I_a F^a_{\mu\nu} \end{aligned} \quad (10)$$

and the Yang-Mills Lagrangian for the gauge field is

$$\mathcal{L}_F = -\frac{1}{4}F^{\mu\nu}_a F^a_{\mu\nu} \; . \quad (11)$$

Since the Yang-Mills field tensor (10) contains the gauge field commutator, which is of second order in

[1] this dimension is referred to as *weak isospin space* for the weak interaction and *color space* for the strong interaction.

the field, the Yang-Mills Lagrangian contains third and fourth order terms. The non-abelian gauge bosons can thus self-interact.

The gauge bosons are necessarily massless (as the mass term breaks gauge invariance). However with the so called Higgs mechanism the gauge bosons can acquire effective mass through interactions with the Higgs field.

The number of gauge bosons is equal to the number of generators in the gauge group. There is one gauge boson, the photon, for the $U(1)$ group; three gauge bosons for the $SU(2)$ group; and eight gauge bosons for the $SU(3)$ group.

Exercises

1. Find the (infinitesimal) gauge-transformation rule for the gauge field A^a_μ.

2. Show that the Yang-Mills field tensor (10) is gauge invariant.

3. (This exercise is actually for the next note!) Show that the fields A, Z, and $W^\pm$ are eigenstates of the charge operator $\hat{Q}$.

   Hints:

   (a) **How to define an abstract operator $\hat{I}$ which represents an infinitesimal generator I:**
   The action of an infinitesimal group element $1 + iI\alpha$, acting on its relevant object ϕ, is by definition $\phi \to (1 + iI\delta\alpha)\phi \equiv 1 + \delta\phi$, where *delta* denotes group covariant differential $\delta\phi = iI\delta\alpha\phi$.

   Thus, apparently, the action of an abstract operator $\hat{I}$, representing the generator I in the space of the object Φ the operator $\hat{I}$ acts upon, can then be defined through the covariant differential $\delta\Phi$ of the object under the infinitesimal group transformation.

$$\hat{I}\Phi = \frac{1}{i}\left.\frac{\delta\Phi}{\delta\alpha}\right|_{\alpha=0} .$$

   (b) **How the abstract operator $\hat{Y}_W$ acts on the gauge fields B and W:**
   Under the gauge transformation generated by the week hyper-charge $U(1)_W$ the covariant part of the transformation of the fields B and W is zero, thus

$$\frac{\hat{Y}_W}{2}B = \frac{\hat{Y}_W}{2}W = 0 .$$

   The fields B and W are thus eigenstates of $\hat{Y}_W$ with the eigenvalue 0.

   (c) **How the abstract operator $\hat{T}_3$ acts on the gauge fields B and W:**
   Under the infinitesimal $SU(2)_L$ gauge transformation,

$$\delta B = 0 \ , \ \delta W = i[\alpha, W] \ ,$$

   where $\alpha = \alpha_a T_a$, $W = W_a T_a$, $T_a = \frac{1}{2}\tau_a$. Thus

$$\hat{T}_3 B = 0 \ ,$$
$$\hat{T}_3 W = [T_3, W]$$

   Then, apparently,

$$\hat{T}_3 W_3 = 0$$
$$\hat{T}_3 W^\pm = \pm W^\pm$$

   where

$$W^\pm = \frac{1}{\sqrt{2}}(W_1 \mp iW_2)$$

   (d) **How the abstract operator $\hat{Q}$ acts on the gauge fields A, Z and $W^\pm$:**
   From the definition $\hat{Q} = \frac{1}{2}\hat{Y}_W + \hat{T}_3$ it immediately follows that

$$\hat{Q}A = 0$$
$$\hat{Q}Z = 0$$
$$\hat{Q}W^\pm = \pm W^\pm$$