Reaction amplitude

An archetypal process in quantum field theory is a reaction where two particles are prepared (e.g. in an accelerator) in an initial plane-wave state with given momenta, for example, a boson with momentum $\mathbf{k}$ and a fermion with momentum $\mathbf{p}$ and polarization λ,

$$|i\rangle = a^\dagger_{\mathbf{k}} a^\dagger_{\mathbf{p}\lambda}|0\rangle \ , \tag{1}$$

where $|0\rangle$ is the vacuum state. The particles are then sent toward each other, a reaction happens with certain probability, and the reaction products are detected by the detection system in their final plane-wave states with certain momenta. Suppose the detectors measure the same particles (elastic scattering) with some (presumably changed in the reaction) momenta and polarization,

$$\langle f| = \langle 0|a_{\mathbf{k}'}a_{\mathbf{p}'\lambda'} \ . \tag{2}$$

The reaction amplitude is given by the matrix element

$$S_{fi} = \langle f|S|i\rangle \tag{3}$$

where the S-matrix is written as a perturbation series of T-products of the interaction Lagrangians

$$S = \sum_n \frac{i^n}{n!}\int \prod_{j=1}^n d^4x_j T \prod_{j=1}^n \mathcal{L}_v(x_j) \ . \tag{4}$$

Normal product

Let us, as an example, take $\mathcal{L}_v = -g\bar{\psi}\psi\phi$ and consider the second order term,

$$S^{(2)} = \frac{(ig)^2}{2!}\int d^4x d^4x' \times$$
$$\times T\Big(\bar{\psi}(x)\psi(x)\phi(x)\bar{\psi}(x')\psi(x')\phi(x')\Big) \ . \tag{5}$$

Apparently, the matrix element (3) could be easily calculated if only instead of the T-product there were an arrangement with all generators to the left of all annihilators (since $a|0\rangle = 0 = \langle 0|a^\dagger$). Such ordering is called “normal” or N-product,

$$N(\text{fields}) = (\text{all generators})(\text{all annihilators}) \ . \tag{6}$$

Propagator

Let us consider the simplest T-product: that of two scalar fields. The difference between T- and N-product of two fields is called a *propagator*

$$\Delta(AB) = T(AB) - N(AB) \ . \tag{7}$$

Apparently, if the fields commute, e.g. $[\phi,\phi] = [\phi^\dagger,\phi^\dagger] = 0$, there is no difference between T- and N-product.

$$\Delta(\phi\phi) = \Delta(\phi^\dagger\phi^\dagger) = 0 \tag{8}$$

Let us calculate the propagator $\Delta(\phi(x)\phi^\dagger(x'))$. Introducing the positive- and negative-frequency waves,

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x) \ , \tag{9}$$

where

$$\phi^{(+)}(x) = \sum_{\mathbf{k}} \frac{a_{\mathbf{k}}e^{-ikx}}{\sqrt{2\omega_{\mathbf{k}}}} \ , \tag{10}$$

$$\phi^{(-)}(x) = \sum_{\mathbf{k}} \frac{a^\dagger_{\mathbf{k}}e^{+ikx}}{\sqrt{2\omega_{\mathbf{k}}}} \ , \tag{11}$$

we get the N-product as

$$\begin{aligned} N\Big(\phi(x)\phi^\dagger(x')\Big) &= \phi^{(+)\dagger}(x')\phi^{(+)}(x) \\ &+\phi^{(+)}(x)\phi^{(-)\dagger}(x') \\ &+\phi^{(-)}(x)\phi^{(+)\dagger}(x') \\ &+\phi^{(-)}(x)\phi^{(-)\dagger}(x') \ . \end{aligned} \tag{12}$$

The T-product is given as

$$T\Big(\phi(x)\phi^\dagger(x')\Big) = \begin{cases} \phi(x)\phi^\dagger(x') \ , & t > t' \\ \phi^\dagger(x')\phi(x) \ , & t < t' \end{cases} . \tag{13}$$

Calculating the difference gives

$$\Delta\Big(\phi(x)\phi^\dagger(x')\Big) = \begin{cases} \Delta^{(+)}(x-x') \ , & t > t' \\ -\Delta^{(-)}(x-x') \ , & t < t' \end{cases} . \tag{14}$$

where

$$\begin{aligned} \Delta^{(\pm)}(x-x') &\equiv [\phi^{(\pm)}(x), \phi^{(\pm)\dagger}(x')] \\ &= \pm\sum_{\mathbf{k}} \frac{e^{\mp ikx}}{2\omega_{\mathbf{k}}} \end{aligned} \tag{15}$$

The propagator is usually written as a Fourier integral,

$$\Delta\Big(\phi(x)\phi^\dagger(x')\Big) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{k^2-m^2+i0} \ . \tag{16}$$

In this form it is easy to see that (i times) the scalar propagator is the Green’s function of the Klein-Gordon equation.

Note that the propagator is not an operator, but a (complex) number. Apparently this happened because the commutator of the (scalar) field operators is a number. For fermionic fields one has to redefine slightly T- and N-products such that their difference, the propagator, will contain an anti-commutator. Specifically,

$$T(\psi(x)\bar\psi(x')) = \left\{ \begin{array}{cc} \psi(x)\bar\psi(x') \; , & t > t' \\ -\bar\psi(x')\psi(x) \; , & t < t' \end{array} \right. . \quad (17)$$

and

$$\begin{aligned} N(\psi(x)\bar\psi(x')) = & - \bar\psi^{(+)}(x')\psi^{(+)}(x) \\ & + \psi^{(+)}(x)\bar\psi^{(-)}(x') \\ & + \psi^{(-)}(x)\bar\psi^{(+)}(x') \\ & + \psi^{(-)}(x)\bar\psi^{(-)}(x') \; . \end{aligned} \quad (18)$$

Analogously, (i times) the bispinor propagator must be the Green's function of the Dirac equation (which can be proved by no so easy calculation)

$$\Delta(\psi_\alpha(x)\bar\psi_\beta(x')) = \int \frac{d^4p}{(2\pi)^4} \left(\frac{i}{\gamma p - m + i0} \right)_{\alpha\beta} e^{-ip(x-x')} \; , \quad (19)$$

where α and β indicate the bispinor components.

For the electromagnetic field the propagator depends on the gauge. One particular choice (Feynman gauge) is

$$\Delta(A_\mu(x)A_\nu(x')) = \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\mu\nu}}{k^2 + i0} e^{-ikx} \; . \quad (20)$$

Wick's theorem

The T-product of fields is equal the sum of normal products of all possible combinations of propagators.

$$\begin{aligned} T(A \ldots Z) = N(A \ldots Z) \\ +\Delta(AB)N(B \ldots Z) \\ +\Delta(AB)\Delta(CD)N(E \ldots Z) \\ \pm(\text{all other combinations of propagators}) \ldots \end{aligned} \quad (21)$$

where $\pm$ indicates that when two anti-commuting operators are commuted (to move an operator next to its propagator partner) a minus sign should be recorded.

For example,

$$T(AB) = N(AB) + \Delta(AB) \quad (22)$$

for two operators and so forth by induction...

Feynman diagrams in coordinate space

A Feynman diagram is a graphical representation of a term in the Wick's expansion of the S-matrix in a given perturbation order,

$$S^{(n)} = \frac{i^n}{n!} \int \prod_{j=1}^{n} d^4x_j T \prod_{j=1}^{n} \mathcal{L}_v(x_j) \; . \quad (23)$$

The diagrams are drawn according to the following rules,

1. Each coordinate x_j is represented by a point;
2. A propagator $\Delta(\phi(x_i)\phi^\dagger(x_j))$ is represented by a broken line connecting points x_i and x_j;
3. A propagator $\Delta(\psi(x_i)\bar\psi(x_j))$ is represented by a solid line connecting points x_i and x_j;
4. A field $\phi(x_i)$ is represented by a broken line attached to the point x_i;
5. A field $\psi(x_i)$ is represented by a solid line attached to the point x_i with an arrow toward the point;
6. A field $\bar\psi(x_i)$ is represented by a solid line attached to the point x_i with an arrow from the point;

The Feynman digrams look a lot nicer in momentum space, but we have got no time for that, I am afraid...

Exercises

1. Show that the S-matrix is unitary, $S^\dagger S = 1$.
2. Show that $\Delta(\phi(x)\phi^\dagger(x')) = \langle 0|T(\phi(x)\phi^\dagger(x')|0\rangle$.
3. Show that the function $i\Delta(\phi(x)\phi(x'))$ is the Green's function of the Klein-Gordon equation.
4. For $\mathcal{L}_v = -g\bar\psi\psi\phi$ apply the Wick's theorem to the second order term of the S-matrix; for each term draw the corresponding Feynman diagram and interpret the term.
5. For $\mathcal{L}_v = -g\bar\psi\psi\phi$ draw the lowest order Feynman diagram of the elastic scattering of two bosons and write down the corresponding term of the S-matrix.