Schwarzschild solution

Schwarzschild solution is the spherically symmetric static solution to the Einstein equation in vacuum. It describes the gravitational field outside a slowly rotating spherical body, like a star, a planet, or a black hole. Karl Schwarzschild has found this solution in 1915 and published it in January 1916 shortly after the publication of Einstein's general theory of relativity.

Schwarzschild metric

Schwarzschild metric is a static and spherically symmetric solution to the vacuum Einstein equation

$$R_{ab} = 0\,. \tag{1}$$

It is written in Schwarzschild coordinates which are realised with a set of static spheres symmetrically nested around the central body. In these coordinates the spherically symmetric static metric can be generally assumed to have the following form,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{2}$$

where A and B are some yet unknown functions of radius r.

Now we have to calculate the Ricci tensor for this metric and and put it into the vacuum equation (1). That will give us the differential equations for A and B which we then solve and hence obtain the metric.

Instead of doing this tedious work manually, one can use the following GNU Maxima script,

```
derivabbrev:true; /* a better notation for derivatives */
load(ctensor); /* load the package to deal with tensors */
ct_coords:[t,r,o,p]; /* our coordiantes: t r theta phi */
depends([A,B],[r]); /* A and B depend on r only */
lg:ident(4); /* set up a 4x4 identity matrix */
lg[1,1]: A; /* g_{tt} = A */
lg[2,2]:-B; /* g_{rr} = -B */
lg[3,3]:-r^2; /* g_{\theta\theta} */
lg[4,4]:-r^2*sin(o)^2; /* g_{\phi\phi} */
cmetric(); /* computes the prerequisites for further calculations */
christof(mcs); /* calculates Christoffel symbols, mcs[b,c,a]=\Gamma^a_{bc} */
ricci(true); /* calculates ric[a,b], the covariant symmetric Ricci tensor */
```

The script calculates analytically the Christoffel symbols[1] and the Ricci tensor[2] for this metric. The vacuum Einstein equation, $R_{ab} = 0$, can then be integrated analytically[3] which gives the famous Schwarzschild metric,

$$ds^2 = \left(1 - \frac{r_g}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{3}$$

[1] $\Gamma^r_{rr} = \frac{1}{2}\frac{B'}{B}\,,\ \Gamma^t_{tr} = \frac{1}{2}\frac{A'}{A}\,,\ \Gamma^r_{tt} = \frac{1}{2}\frac{A'}{B}\,,\ \Gamma^\theta_{\theta r} = \frac{1}{r}\,,\ \Gamma^r_{\theta\theta} = -\frac{r}{B}\,,\ \Gamma^\varphi_{\varphi r} = \frac{1}{r}\,,\ \Gamma^r_{\varphi\varphi} = -\frac{r\sin^2\theta}{B}\,,\ \Gamma^\varphi_{\varphi\theta} = \cot\theta\,,\ \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta\,.$

[2]

$$\begin{aligned} R_{tt} &= \frac{A''}{2B} + \frac{A'}{B}\left(\frac{1}{r} - \frac{B'}{4B} - \frac{A'}{4A}\right), \\ R_{\theta\theta} &= 1 - \left(\frac{r}{B}\right)' - \frac{1}{2}\left(\frac{A'}{A} + \frac{B'}{B}\right)\frac{r}{B}\,, \\ R_{rr} &= -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB}\,. \end{aligned}$$

[3] Making a linear combination $BR_{tt} + AR_{rr} = 0$ gives $A'B + AB' = 0 \ \Rightarrow\ AB = 1 \ \Rightarrow\ \frac{A'}{A} + \frac{B'}{B} = 0$. Then $R_{\theta\theta} = 0$ gives $B = \frac{1}{1-\frac{r_g}{r}}$, $A = 1 - \frac{r_g}{r}$, where r_g is an integration constant.

The integration constant r_g is determined from the Newtonian limit[4], $r_g = 2GM/c^2$, where M is the mass of the central body ($r_g = 2M$ in the units $G = c = 1$). It is called gravitational or Schwarzschild radius. The gravitational radius for the Earth is about 9mm, for the Sun – about 3km.

Motion in the Schwarzschild metric

Massive bodies

Massive bodies move along geodesics, described by the geodesic equation

$$\frac{d}{ds}\left(g_{ab}u^b\right) = \frac{1}{2}g_{bc,a}u^b u^c \,. \tag{4}$$

For $a = t, \theta, \varphi$ the corresponding equations in the Schwarzschild metric (3) are

$$\frac{d}{ds}\left[\left(1-\frac{2M}{r}\right)\frac{dt}{ds}\right] = 0 \ , \tag{5}$$

$$\frac{d}{ds}\left[r^2\frac{d\theta}{ds}\right] = r^2\sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2 \ , \tag{6}$$

$$\frac{d}{ds}\left[r^2\sin^2\theta\frac{d\varphi}{ds}\right] = 0 \ . \tag{7}$$

The r-equation can be conveniently obtained by dividing the Schwarzschild metric (3) by ds^2,

$$1 = \left(1-\frac{2M}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r^2\left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right] \ . \tag{8}$$

Considering equatorial motion ($\theta = \pi/2$), the first three equations can be integrated as

$$\theta = \frac{\pi}{2} \ , \quad r^2\frac{d\varphi}{ds} = J \ , \quad \left(1-\frac{2M}{r}\right)\frac{dt}{ds} = E \ , \tag{9}$$

where J and E are constants of motion related to momentum and energy. The fourth equation then becomes

$$1 = \frac{E^2}{1-\frac{2M}{r}} - \frac{\frac{J^2}{r^4}r'^2}{1-\frac{2M}{r}} - \frac{J^2}{r^2} \ , \tag{10}$$

where $r' \equiv \frac{dr}{d\varphi}$. Traditionally one makes a variable substitution $r = 1/u$,

$$(1-2Mu) = E^2 - J^2u'^2 - J^2u^2(1-2Mu), \tag{11}$$

and then differentiates the equation once. This gives

$$u'' + u = \frac{M}{J^2} + 3Mu^2 \ . \tag{12}$$

In this form the last term is a relativistic correction to the otherwise non-relativistic equation.

Light rays

The rays of light travel along the null-geodesics where $ds^2 = 0$. Consequently instead of ds one needs to use some parameter $d\lambda$ in the geodesic equations $\frac{Dk^a}{d\lambda} = 0$, where $k^a = \frac{dx^a}{d\lambda}$ and also the unity in the left-hand side of equation (8) has to be substituted with zero. This immediately leads to the equation

$$u'' + u = 3Mu^2 \ , \tag{13}$$

which describes the trajectory of a ray of light in the Schwarzschild metric.

In the absence of the central body, $M = 0$ the space becomes flat, and equation (13) turns into equation for a straight line.

[4] $g_{00} \overset{r\to\infty}{\longrightarrow} 1 + \frac{2\phi}{c^2} = 1 - 2\frac{GM}{c^2 r}$

Exercises

1. Consider a non-relativistic (Newtonian) equatorial ($\theta = \pi/2$) motion of a planet with mass m around a star with mass M.

   (a) Argue that the motion is defined by the Lagrangian (in units $G = 1$)

   $$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{mM}{r}\,,$$

   where the dot denotes the temporal derivative, $\dot{r} \mathrel{\hat{=}} \frac{dr}{dt}$.

   (b) Write down the Euler-Lagrange equations,

   $$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}\,,$$

   for $q = r$ and φ.

   (c) Using the first integral $r^2\dot{\varphi} = J$ rewrite the r-equation as an equation for the function $u(\varphi)$, where $u = 1/r$, and compare with (12).

2. Show that in Newtonian mechanics an equatorial ($\theta = \pi/2$) trajectory of a light ray is described by the equation

   $$u'' + u = 0\,,$$

   where $u \mathrel{\hat{=}} \frac{1}{r}$ and $u' \mathrel{\hat{=}} \frac{du}{d\varphi}$.

3. From the expression for the Schwarzschild metric calculate the Christoffel symbols ${\Gamma^r}_{\varphi\varphi}$, ${\Gamma^\varphi}_{r\varphi}$.

4. Derive the radial equation of motion of a massive body in the Schwarzschild metric from the geodesic equation in the form

   $$\frac{du_a}{ds} = \frac{1}{2}g_{bc,a}u^b u^c\ .$$

5. For the Schwarzschild metric calculate the needed Christoffel symbols and then derive the radial equation of motion for a massive body from the geodesic equation in the form

   $$\frac{du^a}{ds} + {\Gamma^a}_{bc}u^b u^c = 0\ .$$

6. (a) Show that a light ray can travel around a massive star in a circular orbit much like a planet. Calculate the radius (in Schwarzschild coordinates) of this orbit. Answer: $r = \frac{3}{2}(2M)$.

   (b) Is this orbit stable against small radial perturbations? Hint: add a small radial perturbation, δu, to the orbit, derive the (first order) equation for the perturbation and find out whether its solutions always remain small.

7. Show that for the equatorial orbit in the Schwarzschild metric the quantity u_φ is conserved (where u^φ is the φ-component of the four-velocity $u^a \mathrel{\hat{=}} dx^a/ds$).

8. Argue that the first integral $\left(1 - \frac{2M}{r}\right)\frac{dt}{ds}$ is indeed determined by the energy of the body. Hint: look at infinity.

9. (not for exam) Build a Maxima script that proves (analytically) that the Schwarzschild metric is indeed a solution to the vacuum Einstein equation. Hint: `factor(ric[1,1]);`.